A new approach of the kinematic fit

Ting-Yang Chen
Department of Physics, Nanjing University, Nanjing, 21008, China

August 28, 1997

Abstract

A new approach of the kinematic fit in event reconstruction is introduced. The method of *regula falsi* is used to solve the constraint equations, instead of the conventional linear approximation. Tested by Monte Carlo data of $J/\psi \to \mu^+ + \mu^-$, the momentum resolutions are improved by this algorithm.

1 Introduction

Some long lived particles such as ${K_s}^0$, Λ, J/ψ etc. are commonly used to identify their mother particles in high energy physics experiments. In the reconstruction of the mother particle, we would like to have a good measurement of its daughters, so the kinematic calculation is usually employed to improve momentum resolutions.

The procedure of the kinematic fit is briefed as follows: With the use of definite hypotheses about the event type and the corresponding conservation laws, a functional of χ^2 is written out. Its minimization is then performed by the Lagrange multiplier method, and a set of equations is obtained. As these are in general not linear, one has to use the numerical method to solve them. The conventional method is the linear approximations, that is to expand the constraint equations to their first derivitives. Unfortunately this method can not get a good approximation, which can be seen graphycally in Fig. 1 in one dimensional case. Another approach, the method of *regula falsi*[1], will always congerge to the solution, if one can find two points which make the equation have opposite signs (see Fig. 1). The former looks pretty and is easy to be generalized into the multidimensional case while the later is not, but the results of the former are hard to control. In this paper we shall show that in some cases the constraint equations can be reduced into one dimension so that the method of *regula falsi* can be applied to get the fitted values. Tested by Monte Carlo data, the momentum resolutions of the J/ψ in $J/\psi \longrightarrow \mu^+ + \mu^-$ has been improved by the new algorithm.

2 Kinematic calculation

Let us recall the kinematic calculation [2, 3] of an event with a stable particle decaying into its daughters. We denote the measured variables by $x_i, i = 1, 2, \ldots, N$, and their error matrix is called G^{-1}. In most cases we can assume that all x_i are uncorrelated, i.e. $(G^{-1})_{i,j} = 0$, for $i{\neq}j$. The assumption of normally distributed variables of x_i's asserts that the best set of variables is that for which the functional

$$\chi^2 = \sum_{i=1}^{N} G_{i,i}(x_i - {x_i}^0)^2 \tag{1}$$

reaches the minimum, here superscript 0 refers to unfitted quantities, and the variable x_i should satisfy the following constraint equations:

$$f_k(x_1, x_2, \ldots, x_N) = 0, \qquad k = 1, 2, \ldots, K. \tag{2}$$

Introducing the Lagrange multipliers $\alpha_k, k = 1, 2, \ldots, K$, we have

$$\chi^2 = \sum_{i=1}^{N} G_{i,i}(x_i - {x_i}^0)^2 + 2\sum_{k=1}^{K} \alpha_k f_k(\mathbf{x}), \qquad k = 1, 2, \ldots, K \tag{3}$$

here $\mathbf{x}$ represents the vector of $(x_1, x_2, \ldots, x_N)$. The minimization of the functional χ^2 can be found by solving the following simultanous equations:

$$\left\{ \begin{array}{ll} \dfrac{\partial \chi^2}{\partial x_i} = 2\{G_{i,i}(x_i - {x_i}^0) + \displaystyle\sum_{k=1}^{K} \alpha_k f_{k,i}(\mathbf{x})\} = 0, & i = 1, 2, \ldots, N \\ \dfrac{\partial \chi^2}{\partial \alpha_k} = 2f_k(\mathbf{x}) = 0, & k = 1, 2, \ldots, K. \end{array} \right. \tag{4}$$

here

$$f_{k,i}(\mathbf{x}) = \frac{\partial f_k(\mathbf{x})}{\partial x_i}$$

Eqs. 4 are in general not linear, so the method of successive substitutions is used to get the fitted values. The iteration goes on as follows: the $(\nu + 1)$-th step can be derived from

$${x_i}^{\nu+1} = {x_i}^0 - \frac{1}{G_{i,i}} \sum_{k=1}^{K} \alpha^{\nu+1} f_{k,i}(\mathbf{x}^\nu), \qquad i = 1, 2, \ldots, N, \tag{5}$$

where $\mathbf{x}^{\nu+1}$ should satisfy the constraint eqs. 2

$$f_k(\mathbf{x}^{\nu+1}) = 0, \qquad k = 1, 2, \ldots, K, \tag{6}$$

and they are nonlinear either. The simplest way to deal with eqs. 6, is the method of linear approximation, that is to expand eqs. 6 to their fisrt derivatives:

$$f_k(\mathbf{x}^{\nu+1}) \simeq f_k(\mathbf{x}^\nu) + \sum_{i=1}^{N} f_{k,i}(\mathbf{x}^\nu)({x_i}^{\nu+1} - {x_i}^\nu) = 0, \qquad k = 1, 2, \ldots, K. \tag{7}$$

Solving the eqs. 5 and eqs. 7 simultanously, one can get the $\mathbf{x}^{\nu+1}$. The process goes on untill some criteria are fulfilled.

As has mentioned above, the approximation of eq. 7 is not a good one and sometimes the $\mathbf{x}^{\nu+1}$ will diverge from its root with this method. If we could reduce the constraint equations into one, that is, to eliminate $K-1$ unknowns $x_j, j = N-K+2, N-K+3, \ldots, N$, from $K-1$ constraint equations $f_k(\mathbf{x}), k = 1, 2, \ldots, K-1$,

$$\left\{\begin{array}{ll} x_{N-K+2} & = y_1(x_1, x_2, \ldots, x_M), \\ x_{N-K+3} & = y_2(x_1, x_2, \ldots, x_M), \\ & \quad\cdots\cdots\cdots, \\ x_N & = y_{K-1}(x_1, x_2, \ldots, x_M), \end{array}\right. \tag{8}$$

here $M = N-K+1$, and there are only M independent variables left. Substituting them into $f_K(\mathbf{x})$, we have

$$F(\mathbf{x}') = f_K(\mathbf{x}', y_1(\mathbf{x}'), y_2(\mathbf{x}'), \ldots, y_{K-1}(\mathbf{x}')) = 0, \tag{9}$$

here $\mathbf{x}' = (x_1, x_2, \ldots, x_M)$, then the iteration equations will be

$$x_i^{\nu+1} = x_i^0 - \frac{1}{G_{i,i}} \alpha^{\nu+1} \frac{\partial F(\mathbf{x}'^{\nu})}{\partial x_i}, \qquad i = 1, 2, \ldots, M. \tag{10}$$

Here the ${x_i}^{\nu+1}$ is the function of $\alpha^{\nu+1}$. Substituting them into eq. 9, instead of eqs. 6, we have

$$F(\alpha^{\nu+1}) = 0. \tag{11}$$

In such case, the method of *regula falsi* can be applied and a better solution of $\alpha^{\nu+1}$ will be obatined.

3 Reduction of constraints

In general there are several constraints. Can they be reduced into one? We will show that in some cases they can. In a J/ψ decay mode of a forward spectrumeter for example

$$J/\psi \longrightarrow \mu^+ + \mu^-.$$

It can be seen that the constraints consist of a definite decay vertex (x_v, y_v, z_v), momentum and energy conservations. If we put the z axis along the beam direction y is vertical upward, a right handed Cartesian coordinate system is then defined. If we denote the momentum vector of J/ψ, μ^+ and μ^- by (p_x, p_y, p_z), (p_{1x}, p_{1y}, p_{1z}) and (p_{2x}, p_{2y}, p_{2z}) respctively, then the track of J/ψ, with its slopes of $a_x = p_x/p_z$ and $a_y = p_y/p_z$ and intercepts of b_x and b_y, is described as

$$x = \frac{p_x}{p_z} \cdot z + b_x, \tag{12}$$

$$y = \frac{p_y}{p_z} \cdot z + b_y, \tag{13}$$

and same for the μ^+ and μ^-. So the constraint equations can be expressed as

$$\begin{cases} f_1 = & \frac{p_{1x}}{p_{1z}} z_v + b_{1x} - x_v = 0, \\ f_2 = & \frac{p_{2x}}{p_{2z}} z_v + b_{2x} - x_v = 0, \\ f_3 = & \frac{p_{1y}}{p_{1z}} z_v + b_{1y} - x_v = 0, \\ f_4 = & \frac{p_{2y}}{p_{2z}} z_v + b_{2y} - x_v = 0, \\ f_5 = & p_x - p_{1x} - p_{2x} = 0, \\ f_6 = & p_y - p_{1y} - p_{2Y} = 0, \\ f_7 = & P_z - p_{1z} - p_{2z} = 0, \\ f_8 = & \sqrt{M_\psi + {p_x}^2 + {p_y}^2 + {p_z}^2} - \sqrt{M_\mu + {p_{1x}}^2 + {p_{1y}}^2 + {p_{1z}}^2} \\ & -\sqrt{M_\mu + {p_{2x}}^2 + {p_{2y}}^2 + {p_{2z}}^2} = 0, \end{cases} \tag{14}$$

here (b_{1x}, b_{1y}) and (b_{2x}, b_{2y}) are intercepts of μ^+ and μ^- tracks, M_ψ and M_μ are masses of J/ψ and $\mu^\pm$ respectively. From f_1 to f_7, it is easy to get

$$\begin{cases} y_1 & = p_{1x} = \frac{p_{1z}}{z_v}(x_v - b_{1x}), \\ y_2 & = p_{1y} = \frac{p_{1z}}{z_v}(y_v - b_{1y}), \\ y_3 & = p_{2x} = \frac{p_{2z}}{z_v}(x_v - b_{2x}), \\ y_4 & = p_{2y} = \frac{p_{2z}}{z_v}(y_v - b_{2y}), \\ y_5 & = p_x = p_{1x} + p_{2x}, \\ y_6 & = p_y = p_{1y} + p_{2y}, \\ y_7 & = p_z = p_{1z} + p_{2z}. \end{cases} \tag{15}$$

There is only one constraint f_8 and nine independent variables, p_{1z}, p_{2z}, x_v, y_v, z_v, b_{1x}, b_{2x}, b_{1y} and b_{2y}, left, so the method of *regula falsi* can be employed. For the simplicity, we denote the constraint equation f_8 as f and the independent variables as $\mathbf{x} = (x_1, x_2, \ldots, x_9) = (p_{1z}, p_{2z}, x_v, y_v, z_v, b_{1x}, b_{2x}, b_{1y}, b_{2y})$, $\mathbf{y} = (y_1, y_2, \ldots, y_7)$. Then the iteration eq. 10 becomes

$$\begin{gathered} {x_i}^{\nu+1} = {x^0}_i - \frac{1}{G_{i,i}} \alpha^{\nu+1} [\frac{\partial f(\mathbf{x}^\nu)}{\partial x_i} + \sum_{k=1}^{7} \frac{\partial f(\mathbf{x}^\nu)}{\partial y_k} \frac{\partial y_k}{\partial x_i}], \\ i = 1, 2, \ldots, 9. \end{gathered} \tag{16}$$

So the ${x_i}^{\nu+1}$ is the function of $\alpha^{\nu+1}$. Substituting ${x_i}^{\nu+1}, i = 1, 2, \ldots, 7$ into the constraint equation f, we have

$$F(\alpha^{\nu+1}) = f(\mathbf{x}^{\nu+1}(\alpha^{\nu+1}), \mathbf{y}(\mathbf{x}^{\nu+1}(\alpha^{\nu+1}))) = 0. \tag{17}$$

Now the problem is how to select the points, α_1 and α_2, which make the function $F(\alpha)$ have the oppsite signs. The search procedure is as follow. Set $\alpha_1 = \alpha^\nu$ and $\alpha_2 = -\alpha^\nu$ to see if they satisfy

$$F(\alpha_1) \cdot F(\alpha_2) < 0, \tag{18}$$

otherwise set $\alpha_2 = 2\alpha_2$ and $\alpha_2 = -2\alpha_2$ and test again. The process goes on if the inequality 18 can be fulfilled, or terminates, if α_2 is greater than a certain value. Usually it will be done in a few steps, because each iteration would not deviate too far away from the previous approximation. So the easy way is to limit the search steps, say five at the most. For the first iteration, we just take $\alpha_1 = 0$ and $\alpha_2 = 1$ and with the same search procedure as before to find out α_1 and α_2 which satisfy the inequality 18. When both α_1 and α_2 are found, we can go on with the method of *regula falsi* as follows: Let α' be the abscisa of the interaction point of the $x-$axis and the chord joining the points $(\alpha_1, F(\alpha_1))$ and $(\alpha_2, F(\alpha_2))$, see Fig. 1, That is

$$\alpha' = \frac{\alpha_1 F(\alpha_2) - \alpha_2 F(\alpha_1)}{F(\alpha_2) - F(\alpha_1)}.$$

If $F(\alpha') = 0$, The process terminates with a zero of $F(\alpha)$. If $F(\alpha')$ has the same sign as $F(\alpha_2)$, replace α_2 with α', otherwise let $\alpha_1 = \alpha'$. The process is then continued to create a sequence of (α_1, α_2) pairs. After a few steps the constraint eq. 17 can well be approximated. We then go back to eqs. 16 and the iteration is going on as usuall.

4 Results

We have generated 1000 events of $J/\psi \to \mu^+ + \mu^-$ to check both linear approximation and our method. For each event we used the same constraints. Here five succesive itrations have been used for each method and with five approximations for the method of *regula false*. The results are shown in Fig. 2a, Fig. 2b and Fig. 2c for p_x, p_y and p_z error distributions respectively. The thick solid line in each plot is the generated events, the dashed line is the one corrected by the linear approximation and the thin solid line is the one corrected by this algorithm respactively. The standard deviations of p_x, p_y and p_z are 25. 59 Mev/c, 51.17 Mev/c and 1023.45 Mev/c for the generated events, 21.17 Mev/c, 29.85 Mev/c and 426.44 Mev/c for the ones by the conventional calculations, and 15.65 Mev/c, 23.45 Mev/c and 264.39 Mev/c by this method.

The CPU time of the new algorithm is about half of the linear approximation in our case, because the costly matrix calculations are avioded.

Acknowledgements

The author wish to express his thanks to our BES colleagues for their valuable discussions. The work is supported by BEPC National Lab of China.

References

[1] Brice Carnaham, H. A. Luther and James O. Wilkers *Applied Numerical Method* 1969 by John Wiley & Sons, Inc.

[2] B. Ronne, CERN 64-13

[3] A. J. Ketikian, E. V. Komissarov, V. S. Kurbatov and I. N. Silin, Nucl. Instr. and Meth. A314(1992)572

[4] Ting-Yang Chen and Nai-Guo Yao, Nucl. Instr. and Meth. A290(1990)390

Figure caption

Fig. 1 The method of linear approximation (dashed line) and the method of *regula falsi.* (thin solid lines)

Fig. 2 Error distributions of p_x, p_y and p_z of the generated events (thick solid lines), corrected by the linear approximation method (dashed lines) and corrected by this method (thin solid lines)