Lund Area Law and Its Application in the R Scan

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Abstract

Area law in Lund model was developed into a Mote Carlo program, which was used in estimate hadronic efficiencies in BEPC/BES R scan.

1 Introduction

Lund string fragmentation hadronization model describes the general semi-classical picture of hadron production. It has been tested well at high energies.

The well-known Monte Carlo simulation packet JETSET is not built in order to describe few-body states (in particular at the few GeV level in e^+e^- annihilation). Prof. Bo Andersson pointed out that in BEPC energy region $(2-5$ GeV), in order to simulate hadron production correctly, we must rewrite the Lund generator in a suitable form in which the extreme high energy approximation conditions are not essential and the event simulations are exclusive instead of inclusive like in JETSET.

In this work we developed the formalism to use the basic Lund model area law directly for Monte Carlo generator LUARLW.

2 String fragmentation in JETSET

The basic hadron production picture is that string fragmentation by a set of new pairs $(q, \bar{q})$ production in quantum vaccum, hadrons form at vertices.

In JETSET, three assumptions are used:

- High energy approximation: the remaining string always has large energy scale;
- Left-right symmetry: fragmentation from q_0 end or $\bar{q}_0$ end are identical.
- Iterative fragmentation

$$\begin{aligned} string &\Rightarrow hadron + remaining \;\; string \\ &\Rightarrow ......many \;\; similar \;\; steps \\ &\Rightarrow till \;\; the \;\; remainder's \;\; mass \;\; reduced \;\; to \;\; a \;\; low \;\; limit. \end{aligned}$$

They lead string fragmentation function taking the unique form

$$f(z) = \frac{N}{z}(1-z)^a exp(-\frac{bm^2}{z}), \tag{1}$$

which was used in JETSET to govern the string fragmentation. In BEPC energy region, the above three assumptions are much roughly, $f(z)$ violates the basic Lund model.

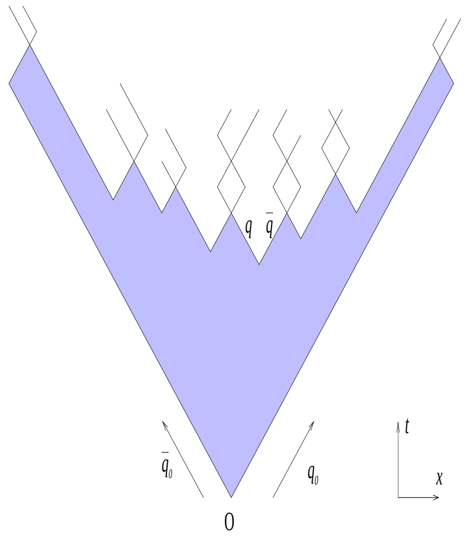

Figure 1: *String fragmentation and mesons production in* $t-x$ *space*

3 The area law in Lund model

The exclusive fragmentation distribution for producing just particular n particles with momentum $\{p_{\circ j}\}$ and nothing else may be written as,

$$dP_n(z_1,\cdots,z_n) = \left[\prod_{k=1}^{n} N_{k-1,k}\frac{dz_k}{z_k}\right]\delta(1-\sum_{k=1}^{n} z_k)\delta(s-\sum_{k=1}^{n}\frac{m_k^2}{z_k})\cdot exp[-bA_n] \quad (2)$$

where, z is the light-cone momentum fraction

$$z = \frac{P_\pm}{P_{0\pm}} = \frac{E \pm P_{//}}{E_0 \pm P_{0//}} \quad (3)$$

and,

$$A_n(z_1,\cdots,z_n) = \sum_{k=1}^{n}\frac{m_k^2}{z_k}\left(\sum_{j=k}^{n} z_j\right). \quad (4)$$

is the area enclosed by quark and antiquark lightcone momentum lines.

The last factor may be viewed as the squared matrix element,

$$|M_n|^2 = exp(-bA_n), \quad (5)$$

the other parts are phase-space elements. The above formula is the area law in Lund model.

Area A satisfys energy-momentum conservation and hadron mass-shell condition,

$$A = A_n(z_1,\cdots,z_n), \quad (6)$$

it corresponds to identity

$$\delta(A - A_n)dA = 1, \tag{7}$$

it may be inserted into the distribution,

$$\begin{aligned} dP_n(z_1, \cdots, z_n) &= \prod_{j=1}^{n} N_{j-1,j} \frac{dz_j}{z_j} \delta(1 - \sum_{j=1}^{n} z_j) \delta(s - \sum_{j=1}^{n} \frac{m_{\perp j}^2}{z_j}) \\ &\cdot \quad \delta(A - A_n) exp(-bA_n) dA. \end{aligned} \tag{8}$$

Finish the integral over $\{z_j\}$, the expression has following form

$$dP_n(s, A) = F_n(s, A)dA. \tag{9}$$

4 Formalisms

(A)String $\Rightarrow$ 2 hadrons
The distribution of string fragmenting into 2 hadrons is

$$\begin{aligned} dP_2 &= N_2 \frac{dz_1}{z_1} \frac{dz_2}{z_2} \delta(1 - z_1 - z_2) \delta(s - \frac{m_{\perp 1}^2}{z_1} - \frac{m_{\perp 2}^2}{z_2}) \\ &\cdot \quad \delta(A - A_2) \cdot exp(-bA) dA. \end{aligned} \tag{10}$$

For the string without gloun,

$$A_2 = \frac{m_{\perp 1}^2}{z_1} + m_{\perp 2}^2, \tag{11}$$

it corresponds to two solutions,

$$z_1^{(1,2)} = \frac{s + m_{\perp 1}^2 - m_{\perp 2}^2 \pm \sqrt{\lambda}}{2s}, \tag{12}$$

$$z_2^{(1,2)} = \frac{s - m_{\perp 1}^2 + m_{\perp 2}^2 \mp \sqrt{\lambda}}{2s}, \tag{13}$$

$$A_2^{(1,2)} = \frac{1}{2}(s + m_{\perp 1}^2 + m_{\perp 2}^2, \mp \sqrt{\lambda}) \tag{14}$$

where

$$\lambda = s^2 + m_{\perp 1}^4 + m_{\perp 1}^4 - 2s(m_{\perp 1}^2 + m_{\perp 2}^2) - 2m_{\perp 1}^2 m_{\perp 2}^2. \tag{15}$$

dP_2 may be represented as the function of area A,

$$dP_2 = \frac{N_2}{\sqrt{\lambda}}[\delta(A - A_2^{(1)}) + \delta(A - A_2^{(2)})]exp(-bA)dA, \tag{16}$$

or

$$P_2 = \frac{N_2}{\sqrt{\lambda}}[exp(-bA_2^{(1)}) + exp(-bA_2^{(2)})]. \tag{17}$$

(B)String $\Rightarrow$ 3 hadrons

The distribution of string fragmenting into 3 hadron is

$$
\begin{aligned}
dP_3 &= N_3 \frac{dz_1}{z_1}\frac{dz_2}{z_2}\frac{dz_3}{z_3}\delta(1-z_1-z_2-z_3)\delta(s-\frac{m_{\perp 1}^2}{z_1}-\frac{m_{\perp 2}^2}{z_2}-\frac{m_{\perp 3}^2}{z_3}) \\
&\cdot \delta(A-A_3)\cdot exp(-bA)dA, \qquad (18)
\end{aligned}
$$

where

$$A_3 = \frac{m_{\perp 1}^2}{z_1} + (1-z_1)\frac{m_{\perp 2}^2}{z_2} + m_{\perp 3}^2. \qquad (19)$$

Integrate over z_1, z_2, z_3,

$$dP_3 = \frac{N_3}{\sqrt{\Lambda}} exp(-bA)dA, \qquad (20)$$

where,

$$
\begin{aligned}
\Lambda &= [(s-A)(A-m_{\perp 1}^2-m_{\perp 2}^2-m_{\perp 3}^2) \\
&- m_{\perp 1}^2 m_{\perp 2}^2 - m_{\perp 2}^2 m_{\perp 3}^2 - m_{\perp 3}^2 m_{\perp 1}^2]^2 - 4s m_{\perp 1}^2 m_{\perp 2}^2 m_{\perp 3}^2. \qquad (21)
\end{aligned}
$$

With Monte Carlo A is obtained. There are two solutions,

$$
\begin{aligned}
z_1^{(1,2)} &= \frac{-B \pm \sqrt{\Lambda}}{2s(A-m_{\perp 2}^2-m_{\perp 3}^2)}, \\
z_2^{(1,2)} &= \frac{m_{\perp 2}^2(1-z_1^{(1,2)})z_1^{(1,2)}}{(A-m_{\perp 3}^2)z_1^{(1,2)}-m_{\perp 1}^2}, \\
z_3^{(1,2)} &= 1-z_1^{(1,2)}-z_2^{(1,2)}, \qquad (22)
\end{aligned}
$$

where,

$$
\begin{aligned}
B &= A(A-m_{\perp 1}^2-m_{\perp 2}^2-m_{\perp 3}^2), \\
&- s(A+m_{\perp 1}^2-m_{\perp 2}^2-m_{\perp 3}^2), \\
&+ m_{\perp 1}^2 m_{\perp 2}^2 + m_{\perp 2}^2 m_{\perp 3}^2 + m_{\perp 3}^2 m_{\perp 1}^2. \qquad (23)
\end{aligned}
$$

(C)String $\Rightarrow$ multi-hadrons ($n \geq 4$)

Area law is Lorentz invariant and factorization. For string $\Rightarrow$ 4, 5, 6 primary hadrons, the string fragmentation may be divided into two subsystems. One is characterized by light-cone fraction $Z_+^{(1)} = Z_1$, and another by $Z_-^{(2)} = Z_2$. By means of Z_1 and Z_2, their invariant masses are

$$s_1 = Z_1(1-Z_2)s, \qquad (24)$$

$$s_2 = Z_2(1-Z_1)s. \qquad (25)$$

The total area is

$$A = A_1 + A_2 + (1-Z_1)(1-Z_2)s, \qquad (26)$$

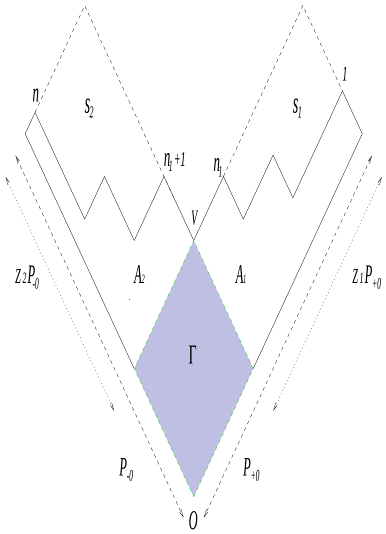

Figure 2: *The vertex V divides the system into two subsystems of "right-movers" (mesons $1, \cdots, n_1$) and "left-movers" (mesons $n_1+1, \cdots, n$).*

where A_1 and A_2 are areas of subsystems.

Assume the subsystems fragment into n_1 and n_2 hadrons, the probability is $(n = n_1 + n_2)$

$$\begin{aligned} dP_n(s) &= dZ_1 dZ_2 ds_1 ds_2 \delta[s_1 - Z_1(1-Z_2)s]\delta[s_2 - Z_2(1-Z_1)s] \\ &\cdot \; exp[-b(1-Z_1)(1-Z_2)s] dP_{n_1}(s_1) \cdot dP_{n_2}(s_2). \end{aligned} \tag{27}$$

Performing the integral over Z_1 and Z_2, the distribution function is represented with s_1 and s_2,

$$dP_n(s) = \frac{ds_1 ds_2}{\sqrt{\lambda(s, s_1, s_2)}} exp(-b\Gamma) P_{n_1}(s_1) P_{n_2}(s_2) \tag{28}$$

5 Multiplicity distributions

We calculate the multiplicity distribution of primary hadrons in the framework of area law. Using single particle transverse mass $m_\perp$ and rapidity y, the momentum may be represented as

$$p_\pm = m_\perp(exp(+y), exp(-y)). \tag{29}$$

The n-particle phase-space factor reads

$$d\Psi = [\prod_{j=1}^{n} \hat{N} dp_j \delta(p_j^2 - m_j^2)] \simeq \frac{\hat{N}^2}{s} \prod_{j=2}^{n-1} \hat{N} dy_j. \tag{30}$$

Assume that the remaining particles are placed in rapidity between $-\Delta y/2$ and $+\Delta y/2$ with

$$\Delta y = ln\frac{s}{s_0}, \tag{31}$$

s_0 is some suitable scale. The phase space factor may be integrated analyticly,

$$\Psi = \int \Psi = \frac{\hat{N}^n (\Delta y)^{n-2}}{s(n-2)!} \tag{32}$$

In case of all particles have the same mass, the fragmentation area A of string way be written as

$$A = m_\perp^2 \sum_{j=1}^{n} \sum_{i=j}^{n} exp(y_i - y_j). \tag{33}$$

Define the dimensionless n-particle partition function

$$\begin{aligned} Z_n &= s \int d\Psi \cdot exp(-bA) \\ &\simeq \frac{(c_1 \Delta y)^n}{n!} exp(-\frac{bm^2 c_2 n^2}{\Delta y}), \end{aligned} \tag{34}$$

which is related to the multiplicity distribution P_n by

$$P_n = \frac{Z_n}{\sum Z_n}. \tag{35}$$

Binominal distribution

$$P(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \tag{36}$$

form a family of distributions depending on the values of parameters N and p. To determine c_1 and c_2 in Z_n from the N and p parameters in binomial distribution, we begin by using Stirlings approximation to write

$$ln(P_n) - ln(P(n)). \tag{37}$$

Express $n = \mu + x$ and expand above formula around $x = 0$ up to terms of order x^2. Equating the series coefficients to zero, we obtain

$$\begin{aligned} c_1 \Delta y &= N\ p\ exp\left[\frac{1 - 2p + 2Np - 2Np^2}{4N^2(1-p)^2}\right], \\ \frac{bm^2 c_2}{\Delta y} &= \frac{1}{2N(1-p)}, \end{aligned} \tag{38}$$

then we have

$$P_n \simeq \frac{(Np)^n}{n!} \cdot exp\left[\frac{-n^2 + 2N\ p\ n}{2N(1-p)}\right]. \tag{39}$$

In practice, we rewrite P_n in a more symetric form

$$P_n = \frac{\mu^n}{n!} \cdot exp[c_0 + c_1(n-\mu) + c_2(n-\mu)^2], \quad (c_0, c_1 > 0, c_2 < 0). \tag{40}$$

The energy-dependence of μ may be expressed as two forms,

$$\mu = a + b \cdot exp(c\sqrt{s}), \tag{41}$$

or

$$\mu = a + b\ ln(s) + c\ ln^2(s). \tag{42}$$

All of the parameters used in above multiplicity distribution have to determined by BES data phenomenologically.

6 Exclusive probability

The probability to obtain a particular state with the n particles is given by

$$\wp_n = C(VPS)(SUD)N^n P_n(m_1, \cdots, m_n; s). \tag{43}$$

The functions $\wp_n$ are the same independent of the rank-ordering of the particles.
- C is the combinatorial number (stemming from the quark contents of the possible strings-there are in general more than one such ordered "quark-string" possible to make the particular mesons and even sometimes the possibility to make the same meson(s) at different places in the same string);
- (VPS) are the vector to pseudoscalar rates;
- (SUD) the strange to up and down probability.
- N is a parameter to obtain the right multiplicity distribution. To obtain the right multiplicity distribution we will sum over all the different orderings and strings in Eq. (43).

7 The transverse momentums schemes

(A) Scheme I

Define the distribution

$$\begin{aligned} & F^{(n)}(\mathbf{p}_{\perp 1}, \cdots, \mathbf{p}_{\perp n}) \\ = \ & \delta(\sum_{j=1}^{n} \mathbf{p}_{\perp j})\theta\left(\sqrt{s} - \sum_{j=1}^{n}\sqrt{m_j^2 + \mathbf{p}_{\perp j}^2}\right) \cdot \prod_{j=1}^{n} exp(-\frac{\mathbf{p}_{\perp j}^2}{2\sigma^2}), \end{aligned} \tag{44}$$

where $\mathbf{p}_{\perp j}$ is a two-dimensional vector, and σ is the variance of the distribution.

The distribution of $\mathbf{p}_{\perp 1}$, $\mathbf{p}_{\perp 2}$, $\cdots$, $\mathbf{p}_{\perp j}$ are

$$\begin{aligned} f_1^{(n)}(\mathbf{p}_{\perp 1}) &= \int d^2\mathbf{p}_{\perp 2} \cdots d^2\mathbf{p}_{\perp n} F^{(n)}(\mathbf{p}_{\perp 1}, \cdots, \mathbf{p}_{\perp n}) \\ &= N_1 \cdot exp[-\frac{n}{n-1}\frac{\mathbf{p}_{\perp 1}^2}{2\sigma^2}], \end{aligned} \tag{45}$$

$$
\begin{aligned}
f_2^{(n)}(\mathbf{p}_{\perp 2}|\mathbf{p}_{\perp 1}) &= \int d^2\mathbf{p}_{\perp 3}\cdots d^2\mathbf{p}_{\perp n}F^{(n)}(\mathbf{p}_{\perp 1},\cdots,\mathbf{p}_{\perp n}) \\
&= N_2 exp[-\frac{n-1}{n-2}\frac{(\mathbf{p}_{\perp 2}+\frac{\mathbf{p}_{\perp 1}}{n-1})^2}{2\sigma^2}]. \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
f_j^{(n)}(\mathbf{p}_{\perp j}|\mathbf{p}_{\perp 1},\cdots,\mathbf{p}_{\perp -1}) &= \int d^2\mathbf{p}_{\perp j+1}\cdots d\mathbf{p}_{\perp n}F^{(n)}(\mathbf{p}_{\perp 1},\cdots,\mathbf{p}_{\perp n}) \\
&= N_j exp[-\frac{(n-j+1)}{(n-j)}\frac{(\mathbf{p}_{\perp j}+\sum_{i=1}^{j-1}\frac{\mathbf{p}_{\perp j}}{n-j+1})^2}{2\sigma^2}]. \qquad (47)
\end{aligned}
$$

The final $\mathbf{p}_{\perp n}$ is determined by energy-momentum conservation. The effective variance of $\mathbf{p}_{\perp j}$ is

$$
\sigma_j^{(eff)} = \sqrt{\frac{n-j}{n-j+1}}\,\sigma, \quad (j=1,\cdots,n-1). \qquad (48)
$$

(B) Scheme II

Particles obtain their transverse momenta from the constituents. At each production point the $(q\bar{q})$-pair is given $\pm\mathbf{q}$ and the particle momenta are

$$
\begin{aligned}
&\mathbf{p}_{\perp 1} = \mathbf{q}_1, \\
&\cdots \\
&\mathbf{p}_{\perp j} = \mathbf{q}_j - \mathbf{q}_{j-1}, \\
&\cdots \\
&\mathbf{p}_{\perp n} = -\mathbf{q}_n. \qquad (49)
\end{aligned}
$$

There is a correlation in the generation of the transverse momentums between adjacent production points. The most general way to introduce such a correlation in a forward-backward symmetric shape is,

$$
\begin{aligned}
F^{(n)}(\mathbf{q}_1,\cdots,\mathbf{q}_n) &= C_n exp\left\{-\frac{1}{2\sigma^2}\left[\mathbf{q}_1^2+\frac{(\mathbf{q}_2-\rho_2\mathbf{q}_1)^2}{1-\rho_2^2}\cdots\right]\right\} \\
&= C_n exp\{-\frac{1}{4\sigma^2}[\mathbf{q}_1^2+\mathbf{q}_n^2+\sum A_j(\mathbf{q}_j^2+\mathbf{q}_{j-1}^2-2\epsilon_j\mathbf{q}_j\cdot\mathbf{q}_{j-1})]\},
\end{aligned}
$$

with

$$
A_j = \frac{(1+\rho_j^2)}{(1-\rho_j^2)}, \qquad \epsilon_j = \frac{2\rho_j}{(1+\rho_j^2)}. \qquad (50)
$$

The correlations ρ_j are phenomenological parameters, which in general are small.

The covariant matrixes

$$
V_2 = \sigma^2\begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix},
$$

$$V_3 = \sigma^2 \begin{pmatrix} 1 & \rho_2 & \rho_2\rho_3 \\ \rho_2 & 1 & \rho_3 \\ \rho_2\rho_3 & \rho_3 & 1 \end{pmatrix},$$

$$V_n = \sigma^2 \begin{pmatrix} 1 & \rho_2 & \cdots & \rho_2\rho_3\cdots\rho_n \\ \rho_2 & 1 & \cdots & \rho_3\cdots\rho_n \\ \cdots & \cdots & \cdots & \cdots \\ \rho_2\cdots\rho_{n-1} & \rho_3\cdots\rho_{n-1} & \cdots & \rho_n \\ \rho_2\cdots\rho_n & \rho_3\cdots\rho_n & \cdots & 1 \end{pmatrix},$$

give the correlation

$$< p_{xi}p_{xj} > \quad and \quad < p_{yi}p_{yj} > . \tag{51}$$

8 Monte Carlo results and BES data

See the spectra.

9 Summary

The Lund model Mote Carlo may be written in two forms:

JETSET (for high energies)
Area Law (for low energies)

In BEPC energy region, we have the following conclusions:
(1) LUARLW can simulate all theoretically permissive initial radiation events, JETSET can not;

(2) LUARLW can generate near-threshold event, JETSET can not;

(3) LUARLW can give normal rapidity correlations, JETSET can not;

(4) LUARLW agrees with all spectrums better than JETSET, espacilly for those which are important and sensitive to hadronic efficiency;

(5) We have reasons believing that LUARLW estimates hadronic efficiencies more reliably, and then reduce the systematic error in R measurement.

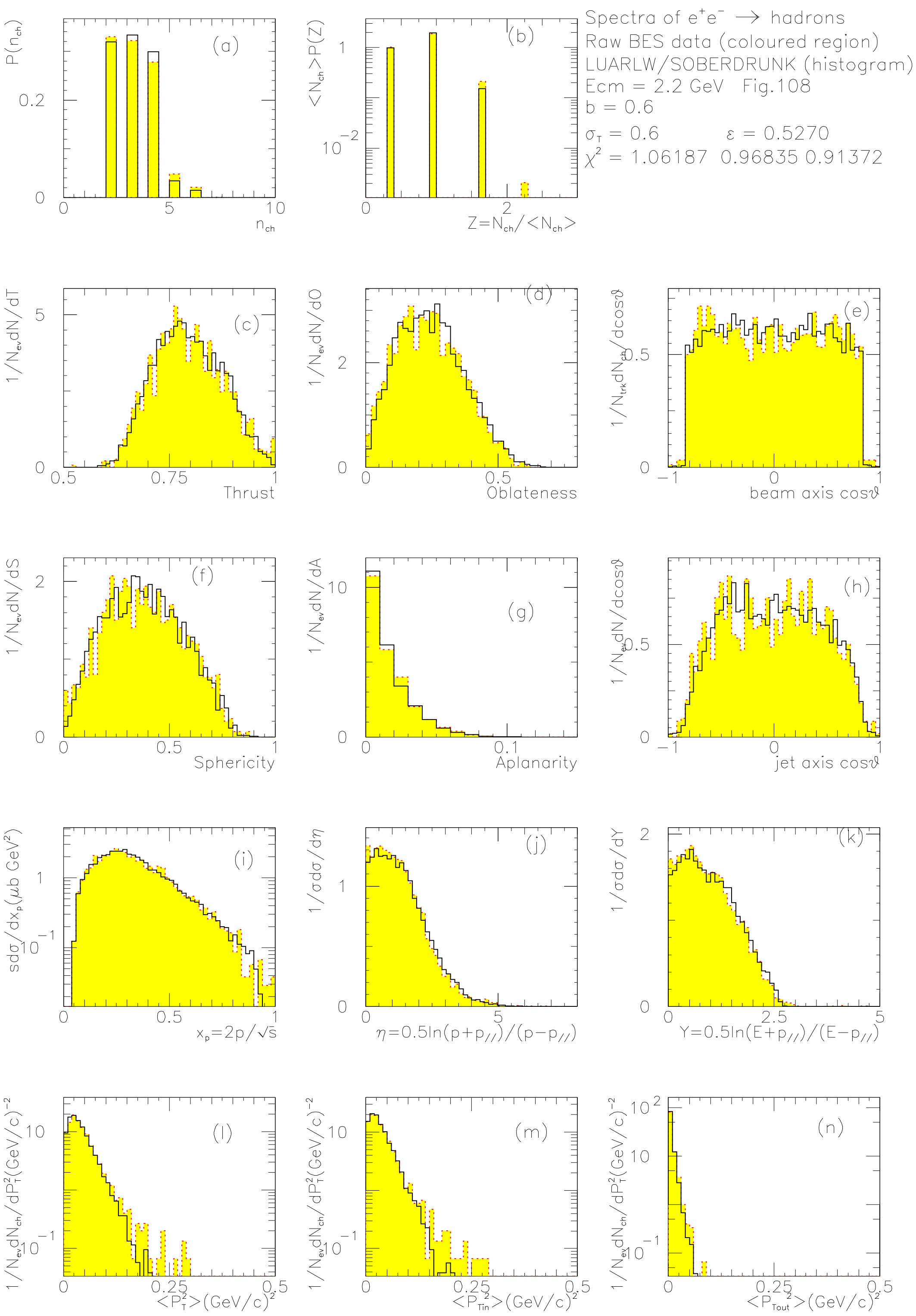

Figure 3: *$e^+e^- \to$ hadrons spectrum of raw BES data (grey region) and LUARLW/SOBDRUNK (black line) at Ecm = 2.2 GeV.*

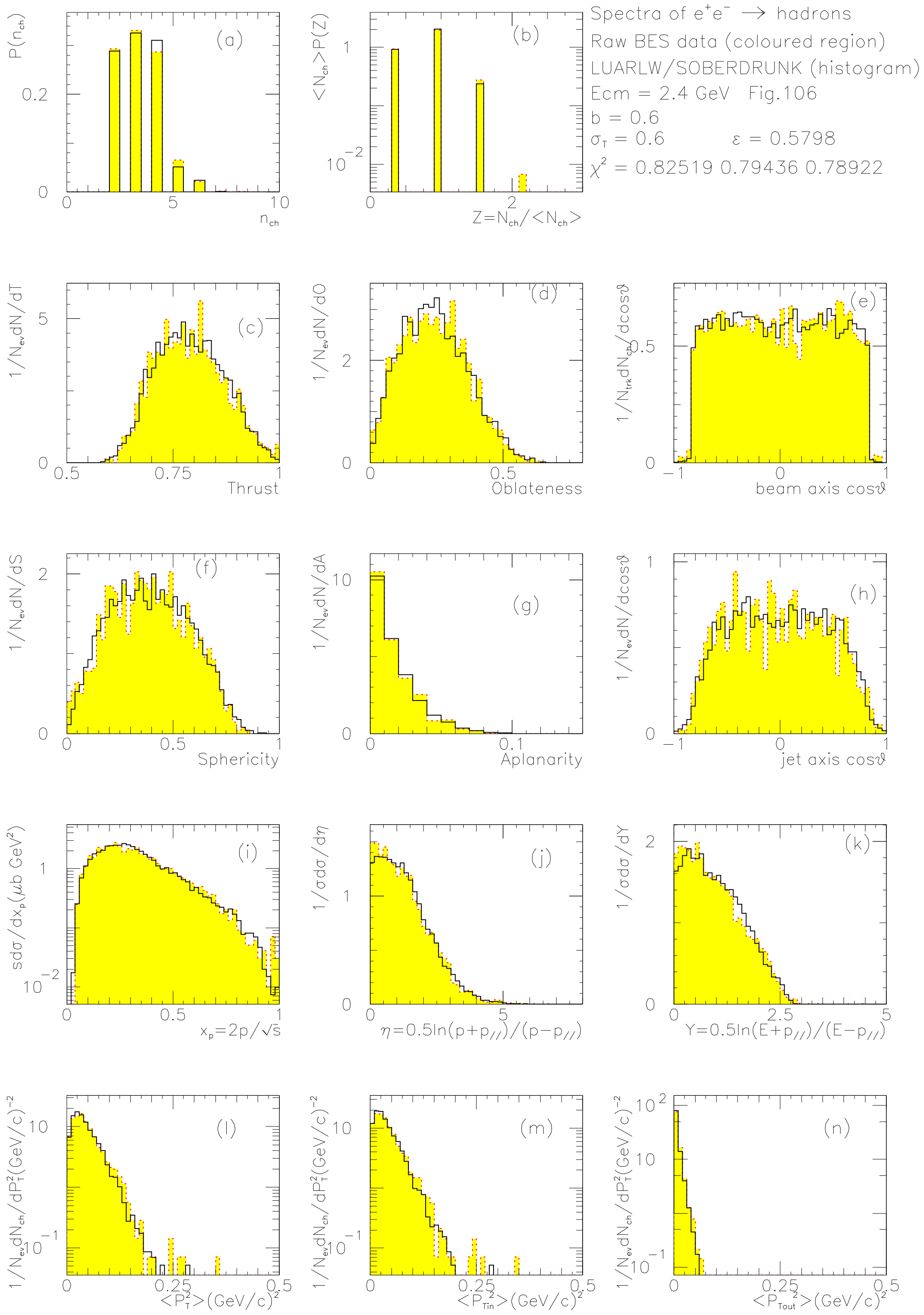

Figure 4: $e^+e^- \rightarrow$ *hadrons spectrum of raw BES data (grey region) and LUARLW/SOBDRUNK (black line) at* $Ecm = 2.6 \quad GeV$.

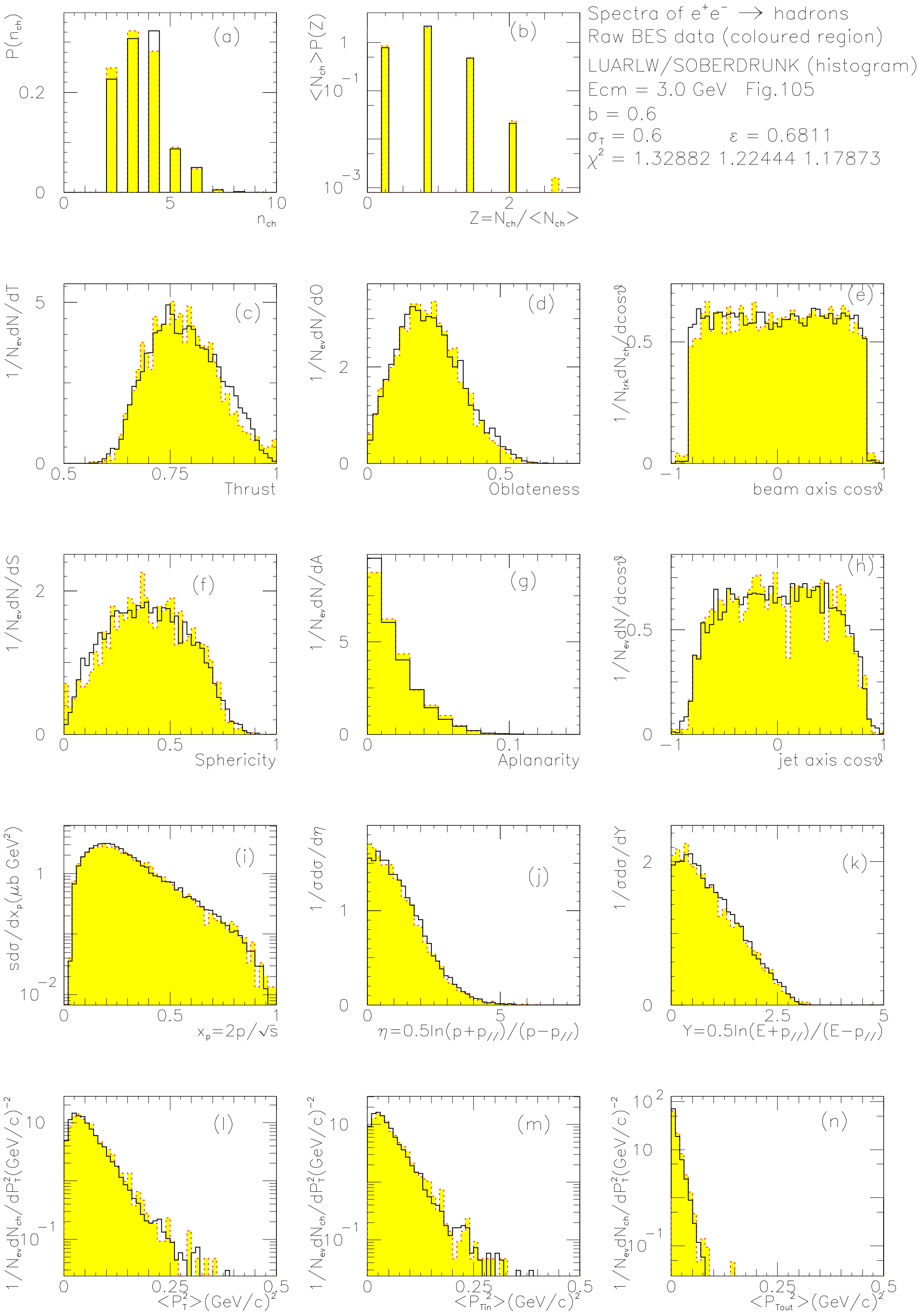

Figure 5: *$e^+e^- \rightarrow$ hadrons spectrum of raw BES data (grey region) and LUARLW/SOBDRUNK (black line) at $Ecm = 3.0$ GeV.*

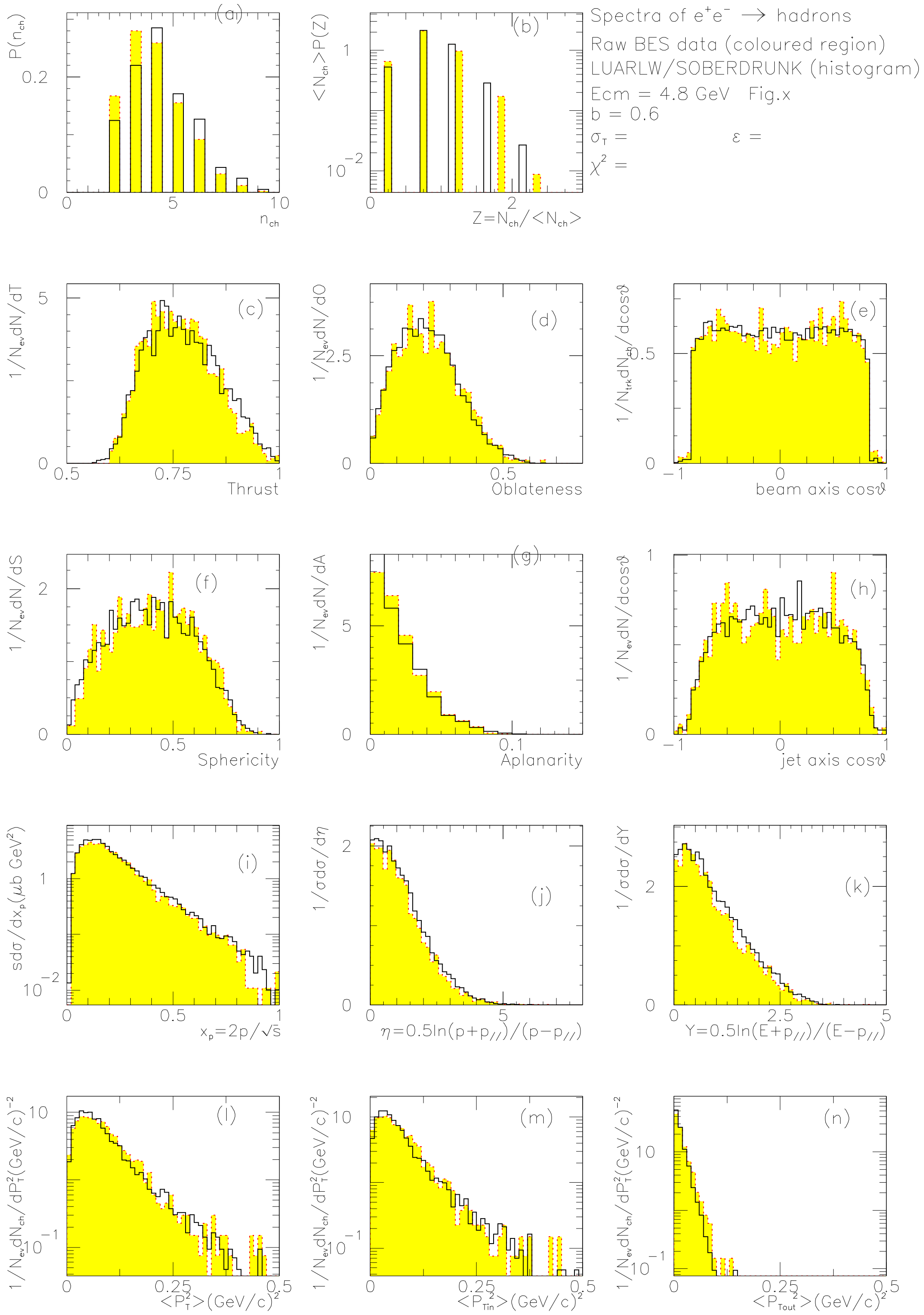

Figure 6: $e^+e^- \rightarrow$ *hadrons spectrum of raw BES data (grey region) and LUARLW/SOBDRUNK (black line) at* $Ecm = 4.8\ GeV$.