An Elementary Introduction to
Partial-Wave Analyses at BNL
—Part II—

S. U. Chung

Physics Department, Brookhaven National Laboratory,

Upton, NY 11973

`http://cern.ch/suchung/`
`http://www.phy.bnl.gov/˜e852/reviews.html`

Plan of Talk

Plan of Talk

- Introduction:

  General References,

  Notations and Conventions

Plan of Talk

- Introduction:

  General References,

  Notations and Conventions

- Reactions: $a + b \to c + d$

  $\pi^- + p \to X^- + p$ and $\pi^- + p \to X^0 + n$

Plan of Talk

- Introduction:

  General References,

  Notations and Conventions

- Reactions: $a + b \to c + d$

  $\pi^- + p \to X^- + p$ and $\pi^- + p \to X^0 + n$

- General n-body Decay Modes of X where $n \geq 3$:

  $X \to \pi + \pi + \pi$, $X \to K + \bar{K} + \pi$,

  $X \to \pi + \pi + \pi + \pi$, $X \to \eta + \pi + \pi + \pi$

Plan of Talk

- Introduction:

  General References,

  Notations and Conventions

- Reactions: $a + b \to c + d$

  $\pi^- + p \to X^- + p$ and $\pi^- + p \to X^0 + n$

- General n-body Decay Modes of X where $n \geq 3$:

  $X \to \pi + \pi + \pi$, $X \to K + \bar{K} + \pi$,

  $X \to \pi + \pi + \pi + \pi$, $X \to \eta + \pi + \pi + \pi$

****

- Simple Decay Modes of X where $n = 2$:

  $X \to \pi + \pi$, $X \to \eta + \pi$

Plan of Talk

- Introduction:

  General References,

  Notations and Conventions

- Reactions: $a + b \to c + d$

  $\pi^- + p \to X^- + p$ and $\pi^- + p \to X^0 + n$

- General n-body Decay Modes of X where $n \geq 3$:

  $X \to \pi + \pi + \pi$, $X \to K + \bar{K} + \pi$,

  $X \to \pi + \pi + \pi + \pi$, $X \to \eta + \pi + \pi + \pi$

****

- Simple Decay Modes of X where $n = 2$:

  $X \to \pi + \pi$, $X \to \eta + \pi$

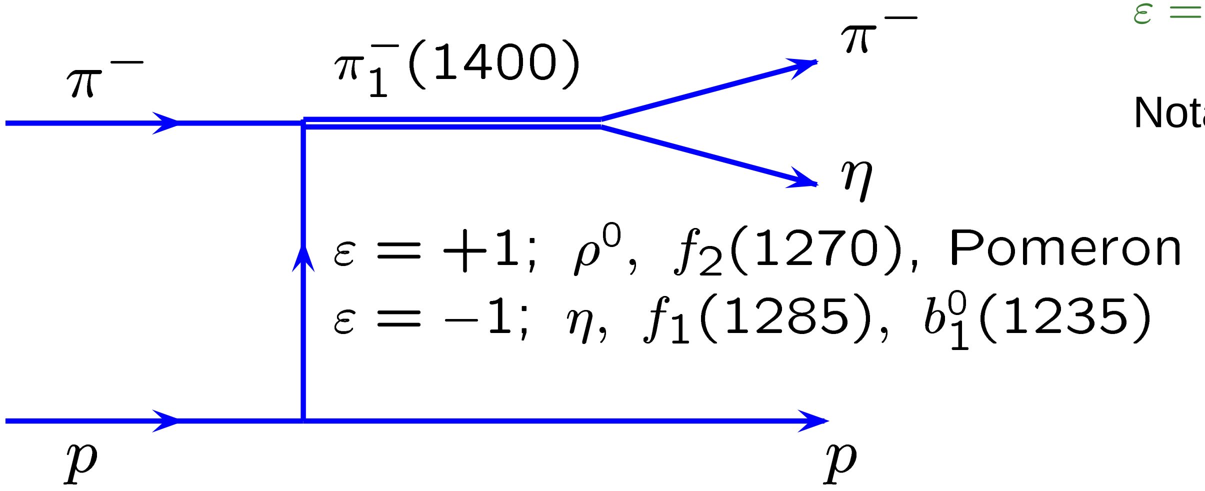
- Conclusions and Future Prospects

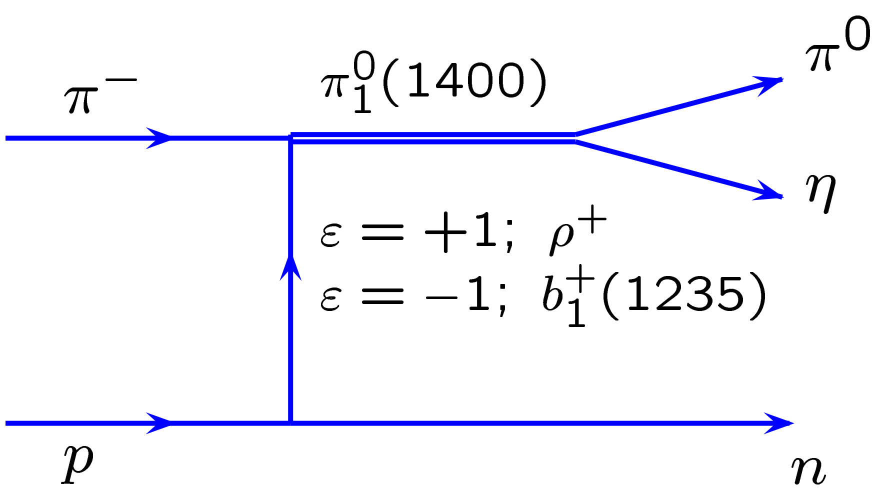
Reggeon exchange:

$\varepsilon = +1$ Natural-parity exchange

$\varepsilon = -1$ Unnatural-parity exchange

Notation: $J^{PC} M^{\varepsilon} R_1 \begin{bmatrix} L \\ S \end{bmatrix} R_2$





$1^{-+}1^{+}\,\eta \begin{bmatrix} P \\ 0 \end{bmatrix} \pi \rightarrow P_+$

$2^{++}1^{+}\,\eta \begin{bmatrix} D \\ 0 \end{bmatrix} \pi \rightarrow D_+$

$1^{-+}0^{-}\,\eta \begin{bmatrix} P \\ 0 \end{bmatrix} \pi \rightarrow P_0$

$1^{-+}1^{-}\,\eta \begin{bmatrix} P \\ 0 \end{bmatrix} \pi \rightarrow P_-$

$2^{++}0^{-}\,\eta \begin{bmatrix} D \\ 0 \end{bmatrix} \pi \rightarrow D_0$

$2^{++}1^{-}\,\eta \begin{bmatrix} D \\ 0 \end{bmatrix} \pi \rightarrow D_-$

Amplitude Analysis for Two-pseudoscalar Systems

Primary References:

S. U. Chung and T. L. Trueman, Phys. Rev. D11, 633 (1975).

S. U. Chung, Phys. Rev. D56, 7299 (1997).

S. U. Chung *et al.*, Phys. Rev. D60, 092001 (1999).

Original References:

E. Barrelet, Nuovo Cimento **8A**, 331 (1972).

S. A. Sadovsky
'On the ambiguities in the partial-wave analysis of $\pi^- p \to \eta\pi^0 n$ Reaction,'
IHEP 91-75 (1991).

General Angular Distributions

Consider the following reaction

$$\pi^- p \rightarrow \pi^0 \eta n$$

In the Jackson frame, the amplitudes may be expanded in terms of the partial waves for the $\pi\eta$ system:

$$U_k(\Omega) = \sum_{\ell m} V_{\ell m k} A_{\ell m}(\Omega)$$

where $V_{\ell m k}$ stands for the production amplitude for a state $|\ell m\rangle$ and k represents the spin degrees of freedom for the initial and final nucleons ($k = 1, 2$ for spin-flip and spin-nonflip amplitudes). $A_{\ell m}(\Omega)$ is the decay amplitude given by

$$A_{\ell m}(\Omega) = \sqrt{\frac{2\ell + 1}{4\pi}}\, D^{\ell\,*}_{m0}(\phi, \theta, 0) = Y^m_\ell(\Omega)$$

where the angles $\Omega = (\theta, \phi)$ describe the direction of the η in the Jackson frame. The angular distribution is given by

$$I(\Omega) = \sum_k |U_k(\Omega)|^2$$

The eigenstates of this reflection operator are

$$|\epsilon\ell m\rangle = \theta(m)\Big\{|\ell m\rangle - \epsilon(-)^m|\ell - m\rangle\Big\}$$

where

$$\begin{aligned}\theta(m) &= \frac{1}{\sqrt{2}}, \quad m > 0 \\ &= \frac{1}{2}, \quad m = 0 \\ &= 0, \quad m < 0\end{aligned}$$

For a positive reflectivity, the $m = 0$ states are not allowed, i.e.

$$|\epsilon\ell 0\rangle = 0, \quad \text{if} \quad \epsilon = +$$

The reflectivity quantum number ϵ has been defined so that it coincides with the naturality of the exchanged particle in Reaction (1).

$${}^{\epsilon}U_k(\Omega) = \sum_{\ell m} {}^{\epsilon}V_{\ell m k}\sqrt{\frac{2\ell+1}{4\pi}}\;{}^{\epsilon}D^{\ell\;*}_{m0}(\phi,\theta,0)$$

and the resulting angular distribution is

$$I(\Omega) = \sum_{\epsilon k} |{}^{\epsilon}U_k(\Omega)|^2$$

The angular distribution may be expanded in terms of the moments $H(LM)$ via

$$I(\Omega) = \sum_{LM} \left(\frac{2L+1}{4\pi}\right) H(LM) D^{L\,*}_{M0}(\phi,\theta,0)$$

so that

$$H(LM) = \int \mathrm{d}\Omega\; I(\Omega)\; D^{L}_{M0}(\phi,\theta,0)$$

$$H(LM) = \sum_{\epsilon k} \sum_{\substack{\ell m \\ \ell' m'}} \left(\frac{2\ell'+1}{2\ell+1}\right)^{1/2} {}^{\epsilon}V_{\ell m k}\, {}^{\epsilon}V^{*}_{\ell' m' k}\, {}^{\epsilon}b(\ell' m' L M \ell m)\; (\ell' 0 L 0|\ell 0)$$

where a new function ${}^{\epsilon}b$ is a sum of Clebsch-Gordan coefficients:

$${}^{\epsilon}b(\ell' m' L M \ell m) = \theta(m')\theta(m)\Big[(\ell' m' L M|\ell m) + (-)^{M}\,(\ell' m' L\, {-M}|\ell m)$$
$$-\,\epsilon\,(-)^{m'}\,(\ell'\, {-m'} L M|\ell m) - \epsilon\,(-)^{m}\,(\ell' m' L M|\ell\, {-m})\Big]$$

Two assumptions: (1) ${}^{\epsilon}V_{\ell m k} = 0$ if $m > 1$ and (2) rank=1, i.e. k=1.
The angular distribution now is

$$I(\Omega) = \frac{1}{4\pi}\left[f_0(\theta) + 2f_1(\theta)\cos\phi + 2f_2(\theta)\cos 2\phi\right]$$

The f-functions are experimentally measurable, as they are completely determined given a set of moments $\{H\}$. Indeed one finds

$$f_M(\theta) = \sum_{L=0}^{2\ell_m}(2L+1)H(LM)d^L_{M0}(\theta)$$

where ℓ_m is the maximum ℓ in the problem. An alternative expression for $I(\Omega)$ as a function of the partial waves

$$I(\Omega) = \frac{1}{4\pi}\left\{|h_0(\theta) + \sqrt{2}h_-(\theta)\cos\phi|^2 + |\sqrt{2}h_+(\theta)\sin\phi|^2\right\}$$

where

$$\left\{ \begin{array}{ll} h_0(\theta) = \sum_{\ell=0}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_0\, d^{\ell}_{00}(\theta), & [\,\ell\,]_0 = {}^{(-)}V_{\ell 0} \\ h_-(\theta) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_-\, d^{\ell}_{10}(\theta), & [\,\ell\,]_- = {}^{(-)}V_{\ell 1} \\ h_+(\theta) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_+\, d^{\ell}_{10}(\theta), & [\,\ell\,]_+ = {}^{(+)}V_{\ell 1} \end{array} \right.$$

Note that

$$h_0(-\theta) = +h_0(\theta) \quad \text{and} \quad h_\pm(-\theta) = -h_\pm(\theta)$$

Comparing the two expressions for $I(\Omega)$, one finds

(0)
$$\left\{ \begin{array}{l} f_0(\theta) = |h_0(\theta)|^2 + |h_-(\theta)|^2 + |h_+(\theta)|^2 \\ f_1(\theta) = \sqrt{2}\,\mathrm{Re}\{h_0(\theta) h_-^*(\theta)\} \\ f_2(\theta) = \frac{1}{2}\left\{ |h_-(\theta)|^2 - |h_+(\theta)|^2 \right\} \end{array} \right.$$

Define

$$f_a(\theta) \equiv f_0(\theta) + 2f_2(\theta) = |h_0(\theta)|^2 + |\sqrt{2}\,h_-(\theta)|^2$$

$$f_b(\theta) \equiv 2f_1(\theta) = 2\mathrm{Re}\{h_0(\theta)\sqrt{2}\,h_-^*(\theta)\}$$

The form of f_a and f_b suggests that one can define and find

$$\begin{cases} f_a(\theta) = |g(\theta)|^2 + |g(-\theta)|^2 \\ f_b(\theta) = |g(\theta)|^2 - |g(-\theta)|^2 \end{cases} \quad \text{where} \quad \begin{cases} g(\theta) = \dfrac{1}{\sqrt{2}}\left[h_0(\theta) + \sqrt{2}\,h_-(\theta)\right] \\ g(-\theta) = \dfrac{1}{\sqrt{2}}\left[h_0(\theta) - \sqrt{2}\,h_-(\theta)\right] \end{cases}$$

With $u = \tan(\theta/2)$ and $e^\ell_{mm'}(u) = (1+u^2)^\ell\, d^\ell_{mm'}(u)$, we see that

$$(1+u^2)^{\ell_m}\, h_0(u) = \sum_{\ell=0}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_0\,(1+u^2)^{\ell_m-\ell}\, e^\ell_{00}(u)$$

$$(1+u^2)^{\ell_m}\, h_-(u) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_-\,(1+u^2)^{\ell_m-\ell}\, e^\ell_{10}(u)$$

$$(1+u^2)^{\ell_m}\, h_+(u) = \sum_{\ell=1}^{\ell_m} \sqrt{2\ell+1}\,[\,\ell\,]_+\,(1+u^2)^{\ell_m-\ell}\, e^\ell_{10}(u)$$

Suppose now that a set of $[\,\ell\,]$ has been found. One can then find $2\ell_m$ roots of the function

$$(1+u^2)^{\ell_m}\, g(u) = c_0 \prod_{k=1}^{2\ell_m} (u-u_k)$$

where u_k's are complex roots—these are the so-called 'Barrelet' zeroes—and c_0 is a complex constant. Next solve for h's in terms of g's

$$\begin{cases} h_0(\theta) = \dfrac{1}{\sqrt{2}}\Big[g(\theta)+g(-\theta)\Big] \\ h_-(\theta) = \dfrac{1}{2}\Big[g(\theta)-g(-\theta)\Big] \end{cases}$$

and find $|h_+(\theta)|^2 = |h_-(\theta)|^2 - 2f_2(\theta)$

The ambiguity problem for $[\,\ell\,]_+$ can be dealt with by setting

$$(1+u^2)^{\ell_m}\, h_+(u) = c_+\, u \prod_{k=1}^{\ell_m - 1} (u^2 - r_k)$$

where r_k's are the complex roots in u^2 and c_+ is a complex constant. For $\ell_m > 1$, there must be in general $2^{\ell_m - 2}$ solutions for the partial waves with natural-parity exchange, i.e. $[\,\ell\,]_+$.

The total number of ambiguous solutions is, for $\ell_m \leq 4$,

ℓ_m	0	1	2	3	4
N_a	1	2	8	64	512

An Example with S-, P- and D-waves

Consider an example of the $\pi\eta$ system with $\ell_m = 2$, produced in Reaction (1).

$$H(00) = S_0^2 + P_0^2 + P_-^2 + D_0^2 + D_-^2 + P_+^2 + D_+^2$$

$$H(10) = \frac{1}{\sqrt{3}} S_0 P_0 + \frac{2}{\sqrt{15}} P_0 D_0 + \frac{1}{\sqrt{5}} (P_- D_- + P_+ D_+)$$

$$H(11) = \frac{1}{\sqrt{6}} S_0 P_- + \frac{1}{\sqrt{10}} P_0 D_- - \frac{1}{\sqrt{30}} P_- D_0$$

$$H(20) = \frac{1}{\sqrt{5}} S_0 D_0 + \frac{2}{5} P_0^2 - \frac{1}{5} (P_-^2 + P_+^2) + \frac{2}{7} D_0^2 + \frac{1}{7} (D_-^2 + D_+^2)$$

$$H(21) = \frac{1}{\sqrt{10}} S_0 D_- + \frac{1}{5} \sqrt{\frac{3}{2}} P_0 P_- + \frac{1}{7\sqrt{2}} D_0 D_-$$

$$H(22) = \frac{1}{5} \sqrt{\frac{3}{2}} (P_-^2 - P_+^2) + \frac{1}{7} \sqrt{\frac{3}{2}} (D_-^2 - D_+^2)$$

and

$$H(30) = \frac{3}{7\sqrt{5}}(\sqrt{3}P_0 D_0 - P_- D_- - P_+ D_+)$$

$$H(31) = \frac{1}{7}\sqrt{\frac{3}{5}}(2P_0 D_- + \sqrt{3}P_- D_0)$$

$$H(32) = \frac{1}{7}\sqrt{\frac{3}{2}}(P_- D_- - P_+ D_+)$$

$$H(40) = \frac{2}{7}D_0^2 - \frac{4}{21}(D_-^2 + D_+^2)$$

$$H(41) = \frac{1}{7}\sqrt{\frac{5}{3}}D_0 D_-$$

$$H(42) = \frac{\sqrt{10}}{21}(D_-^2 - D_+^2)$$

One should note that the moments $H(4M)$ have contributions from the D-wave only, while the moments $H(3M)$ result from interference between P- and D-waves.

Suppose now that one has found a set of solutions $\{S_0, P_0, P_-, D_0, D_-\}$ for unnatural-parity exchange and $\{P_+, D_+\}$ for natural-parity exchange. It is helpful to write down the h's explicitly:

$$
(0)\quad \begin{cases} (1+u^2)^2\, h_0(u) = S_0\,(1+u^2)^2 + \sqrt{3}P_0\,(1-u^4) + \sqrt{5}D_0\,(1-4u^2+u^4) \\ \sqrt{2}(1+u^2)^2\, h_-(u) = -2u\Big[\sqrt{3}P_-\,(1+u^2) + \sqrt{15}D_-\,(1-u^2)\Big] \\ \sqrt{2}(1+u^2)^2\, h_+(u) = -2u\Big[\sqrt{3}P_+\,(1+u^2) + \sqrt{15}D_+\,(1-u^2)\Big] \end{cases}
$$

The last equation above shows that there are no ambiguities for the partial waves P_+ and D_+, since the expression inside the square bracket is linear in u^2. On the other hand, from the first two equations, one finds that the function $g(u)$ is given by

$$
\begin{aligned} G(u) \equiv \sqrt{2}(1+u^2)^2\, g(u) = S_0\,(1+u^2)^2 + \sqrt{3}P_0\,(1-u^4) + \sqrt{5}D_0\,(1-4u^2+u^4) \\ - 2\sqrt{3}P_-\,(u+u^3) - 2\sqrt{15}D_-\,(u-u^3) \end{aligned}
$$

which is a polynomial of order 4 in u and thus gives rise to the ambiguities in the unnatural-parity partial-waves through the Barrelet zeroes.

One may write

$$G(u) = a_4\, u^4 - a_3\, u^3 + a_2\, u^2 - a_1\, u + a_0$$

with

$$\begin{cases} a_4 = S_0 - \sqrt{3}P_0 + \sqrt{5}D_0 \\ a_3 = 2\sqrt{3}(P_- - \sqrt{5}D_-) \\ a_2 = 2S_0 - 4\sqrt{5}D_0 \\ a_1 = 2\sqrt{3}(P_- + \sqrt{5}D_-) \\ a_0 = S_0 + \sqrt{3}P_0 + \sqrt{5}D_0 \end{cases}$$

The inverse is

$$\begin{cases} 6S_0 = 2a_0 + a_2 + 2a_4 \\ 2\sqrt{3}P_0 = a_0 - a_4 \\ 6\sqrt{5}D_0 = a_0 - a_2 + a_4 \\ 4\sqrt{3}P_- = a_1 + a_3 \\ 4\sqrt{15}D_- = a_1 - a_3 \end{cases}$$

Since $G(u)$ is a 4th-order polynomial in u with 4 complex roots $\{u_1, u_2, u_3, u_4\}$, it is given by

$$G(u) = a_4(u - u_1)(u - u_2)(u - u_3)(u - u_4)$$

so that

$$a_3 = a_4(u_1 + u_2 + u_3 + u_4)$$

$$a_2 = a_4(u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4)$$

$$a_1 = a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2)$$

$$a_0 = a_4(u_1u_2u_3u_4)$$

Finally, the partial waves can be expressed in terms of the roots or the Barrelet zeroes:

$$6S_0 = a_4(2u_1u_2u_3u_4 + u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_2u_4 + u_3u_4 + 2)$$

$$2\sqrt{3}P_0 = a_4(u_1u_2u_3u_4 - 1)$$

$$6\sqrt{5}D_0 = a_4(u_1u_2u_3u_4 - u_1u_2 - u_1u_3 - u_1u_4 - u_2u_3 - u_2u_4 - u_3u_4 + 1)$$

$$4\sqrt{3}P_- = a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 + u_1 + u_2 + u_3 + u_4)$$

$$4\sqrt{15}D_- = a_4(u_1u_2u_3 + u_2u_3u_4 + u_3u_4u_1 + u_4u_1u_2 - u_1 - u_2 - u_3 - u_4)$$

There should be in general 8 ambiguous solutions involving the partial waves S_0, P_0, P_-, D_0 and D_-. The 8 solutions are enumerated below in two columns:

$$\begin{array}{ll}
\{u_1, u_2, u_3^*, u_4^*\} & \{u_1, u_2, u_3^*, u_4\} \\
\{u_1, u_2, u_3, u_4\} & \{u_1, u_2, u_3, u_4\} \\
\{u_1, u_2, u_3, u_4^*\} & \{u_1^*, u_2, u_3, u_4\} \\
\{u_1, u_2, u_3^*, u_4\} & \{u_1, u_2^*, u_3, u_4\} \\
\{u_1, u_2, u_3^*, u_4^*\} & \{u_1, u_2, u_3^*, u_4\} \\
\{u_1, u_2^*, u_3, u_4\} & \{u_1, u_2, u_3, u_4^*\} \\
\{u_1, u_2^*, u_3, u_4^*\} & \{u_1^*, u_2^*, u_3, u_4\} \\
\{u_1, u_2^*, u_3^*, u_4\} & \{u_1^*, u_2, u_3^*, u_4\} \\
\{u_1, u_2^*, u_3^*, u_4^*\} & \{u_1^*, u_2, u_3, u_4^*\}
\end{array}$$

The first column results from a procedure in which u_1 is left invariant and the remaining three roots u_2, u_3 and u_4 are allowed to undergo complex conjugation—one sees that there are 2^3=8 ways of doing this.

Reaction: $\pi^- p \to \eta\pi^- p$ at 18 GeV/c, $\eta \to \gamma\gamma$, $\sigma(\eta \to \gamma\gamma) \sim 30$ MeV
$\sim 47\,200$ events

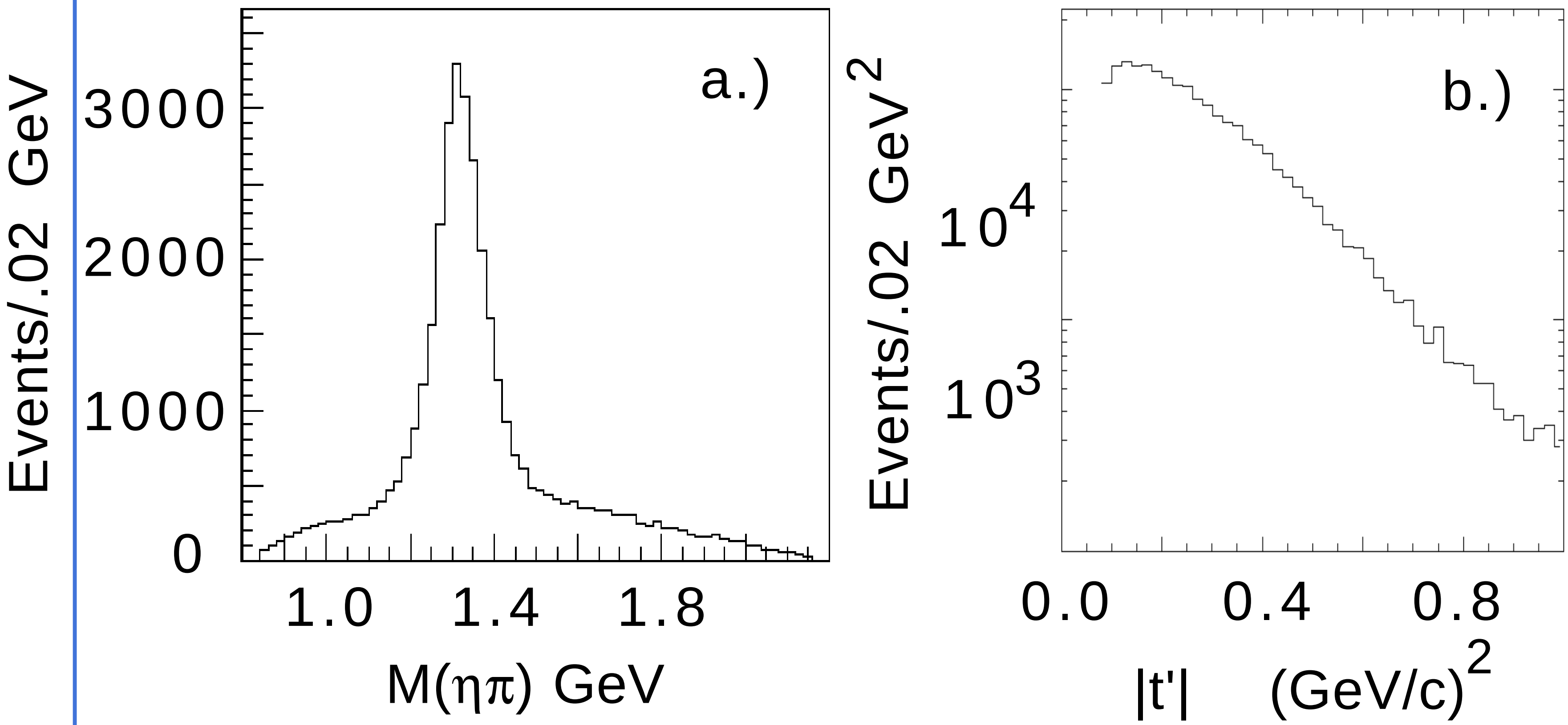
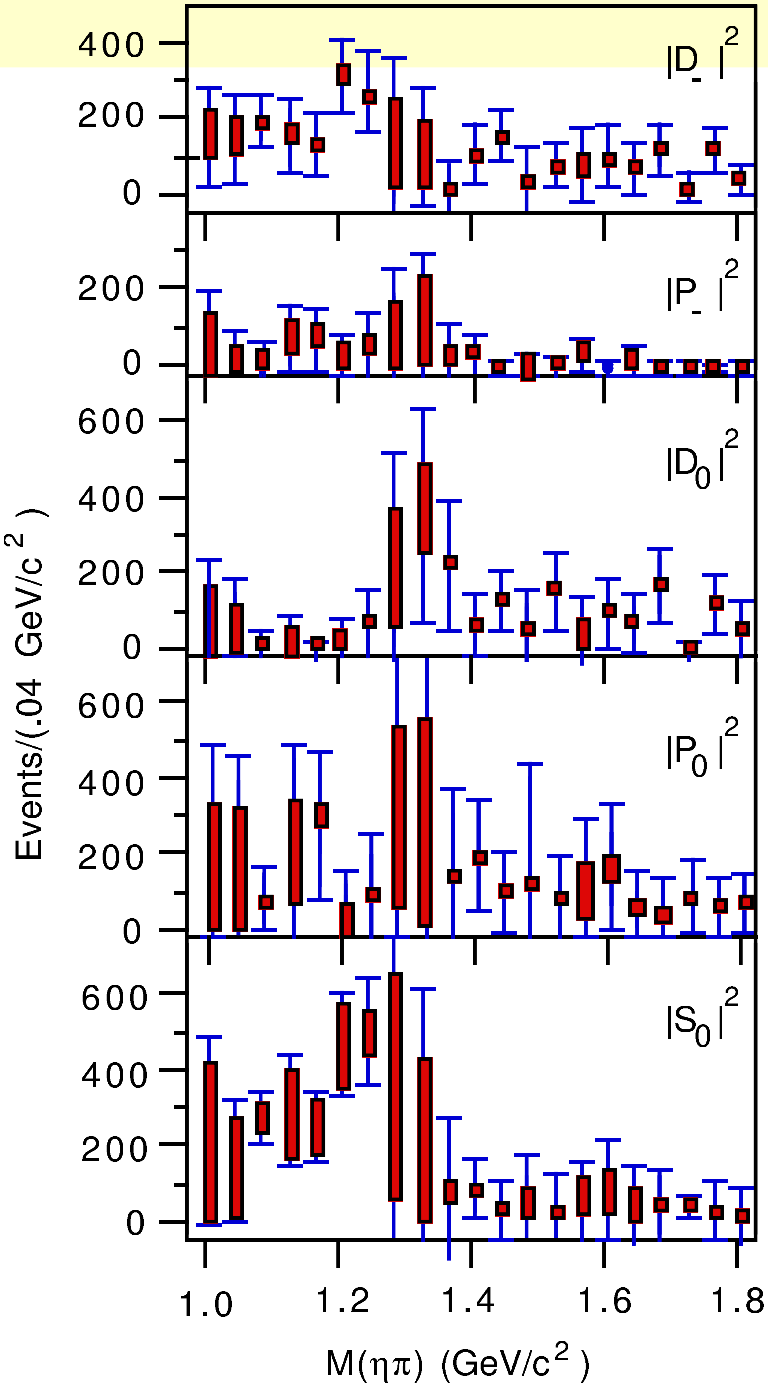


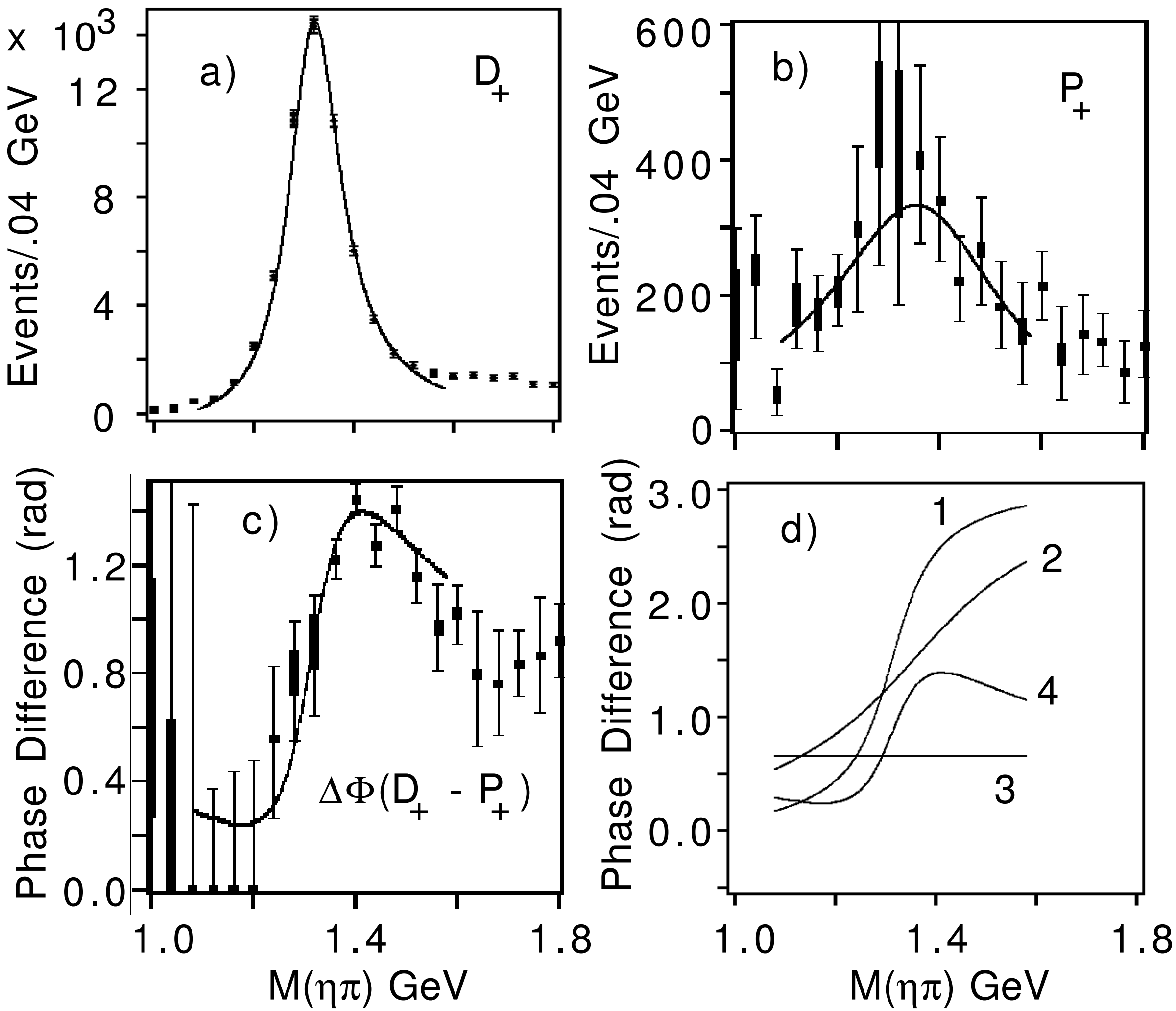
Figure 1

BNL-E852



|D-|^2
|P-|^2
|D0|^2
|P0|^2
|S0|^2
400
200
0
600
Events/(.04 GeV/c^2)
1.0
1.2
1.4
1.6
1.8
M(ηπ) (GeV/c^2)

BNL-E852



$$\begin{cases} M(P_+) = 1370 \pm 16 {}^{+50}_{-30} \\ \Gamma(P_+) = 385 \pm 40 {}^{+\ 65}_{-105} \end{cases}$$

PRL <u>79</u>, 1630 (1997)

PRD <u>60</u>, 092001 (1999)

Decay Amplitudes with t' Dependence

Consider a process

$$a + b \longrightarrow c + d, \qquad c \longrightarrow 1 + 2$$

The usual variable t' is defined through

$$t' = 2\, p_a\, p_c\, (1 - \cos\theta_0)$$

where p_a, p_c and $\cos\theta_0$ are given in the overall center-of-mass (CM) frame. The decay amplitude evaluated in the c rest frame (the c RF) is given by

$$A^{\xi}_{\lambda_1\,\lambda_2}(m,\Omega) = \sqrt{\frac{2j+1}{4\pi}}\, F^{\xi}_{\lambda_1\,\lambda_2}\, D^{j\,*}_{m\,\delta}(\phi,\theta,0), \quad \delta = \lambda_1 - \lambda_2$$

where $\xi = \{j,\eta\}$ with η standing for the intrinsic parity of c and $|jm\rangle$ is the spin state for c before its decay, and $\Omega = (\theta,\phi)$ are the appropriate decay angles of the particle 1 in the c RF.

If $\theta_0 = \epsilon \longrightarrow 0$, then we have

$$t' = 2\, p_a\, p_c\, (1 - \cos\theta_0) \longrightarrow p_a\, p_c\, \epsilon^2$$

Since the amplitude contains a phase factor

$$\exp[i\, m\phi] = \Big(\exp[\pm i\, \phi]\Big)^{|m|}$$

we set

$$\Big(\exp[\pm i\, \phi]\Big)^{|m|} \longrightarrow \Big(\epsilon\, \exp[\pm i\, \phi]\Big)^{|m|} \propto \Big(t'\Big)^{|m|/2} \Big(\exp[\pm i\, \phi]\Big)^{|m|}$$

so that the amplitudes remain stable as $\epsilon \longrightarrow 0$. we introduce a new function of t'

$$Q_m(t') = \Big(v\Big)^{|m|/2} \exp(1 - v), \quad v = \frac{t'}{2t'_0}$$

where t'_0 is a parameter to be determined experimentally. The functions are normalized such that $Q_m(t'_0) = 1$. The modified decay amplitudes are

$$A^{\xi}_{\lambda_1\,\lambda_2}(t', m, \Omega) = \sqrt{\frac{2j+1}{4\pi}}\, Q_m(t')\, F^{\xi}_{\lambda_1\,\lambda_2}\, D^{j\,*}_{m\,\delta}(\phi, \theta, 0)$$

Conclusions and Future Prospects

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

- More systematic program to handle the ambiguities for 2-body decay—including the treatment of identical particles

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

- More systematic program to handle the ambiguities for 2-body decay—including the treatment of identical particles

- Consistent treatment of the 'marginal' partial waves?
  How do we establish the number of partial waves required in a given mass bin?

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

- More systematic program to handle the ambiguities for 2-body decay—including the treatment of identical particles

- Consistent treatment of the 'marginal' partial waves?
  How do we establish the number of partial waves required in a given mass bin?

- Is the isobar model adequate at all times?

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

- More systematic program to handle the ambiguities for 2-body decay—including the treatment of identical particles

- Consistent treatment of the ‘marginal’ partial waves?
  How do we establish the number of partial waves required in a given mass bin?

- Is the isobar model adequate at all times?

- Establish an international group to study/distribute major partial-wave programs

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

- More systematic program to handle the ambiguities for 2-body decay—including the treatment of identical particles

- Consistent treatment of the 'marginal' partial waves?
  How do we establish the number of partial waves required in a given mass bin?

- Is the isobar model adequate at all times?

- Establish an international group to study/distribute major partial-wave programs

- How often do we need to meet in a workshop such as this?

Conclusions and Future Prospects

- Rank n vs rank $(n+1)$:
  When can it be shown that the rank-$(n+1)$ partial waves are not necessary?

- More systematic program to handle the ambiguities for 2-body decay—including the treatment of identical particles

- Consistent treatment of the ‘marginal’ partial waves?
  How do we establish the number of partial waves required in a given mass bin?

- Is the isobar model adequate at all times?

- Establish an international group to study/distribute major partial-wave programs

- How often do we need to meet in a workshop such as this?

- Any volunteers to write a Physics Report on

  the art and craft of partial-wave analyses ?