An Elementary Introduction to

Partial-Wave Analyses at BNL

—Part I—

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http://cern.ch/suchung/
http://www.phy.bnl.gov/~e852/reviews.html

Plan of Talk

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- Introduction:

  General References,

  Notations and Conventions

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- Reactions: $a + b \to c + d$

  $\pi^- + p \to X^- + p$ and $\pi^- + p \to X^0 + n$

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- General n-body Decay Modes of X where $n \geq 3$:

  $X \to \pi + \pi + \pi$, $X \to K + \bar{K} + \pi$,

  $X \to \pi + \pi + \pi + \pi$, $X \to \eta + \pi + \pi + \pi$

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- Simple Decay Modes of X where $n = 2$:

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- Conclusions and Future Prospects

Introduction

General References: Poincaré Group to Rotation Group

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- S. Weinberg:
  'The Quantum Theory of Fields,' Cambridge, UK (1995), Volume I, Chapter 2:
  Relativistic Quantum Mechanics, p.49—
  The Poincaré Algebra
  One-Particle States (Mass> 0 and $= 0$)
  P-, T-, C-Operators
  Michele Maggiore:
  'A Modern Introduction to Quantum Field Theory,'
  Chapter 2 (Lorentz and Poincaré Symmetries in QFT)
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- M. E. Rose, 'Angular Momentum. . . ,' Wiley, NY (1957)
  - Clebsch-Gordan coefficients — my notation: $(j_1 m_1 \; j_2 m_2 | jm)$
  - The d-functions. . .

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  - The d-functions. . .

- S. U. Chung, 'Spin Formalisms,' CERN 71-8 (Updated Version)—spinfm1.pdf
  - Zemach amplitudes. . .

Generators of the Poincaré Group

$$P^\mu, \quad J^{\mu\nu} \to K^\mu \oplus J^\mu$$

Derived quantities:

$$W^\mu \to \quad \vec{S} \quad \text{and} \quad \vec{L}$$

A. J. Macfarlane, J. Math. Phys. **4**, 490 (1963)

A. McKerrell, NC **34**, 1289 (1964)

S. U. Chung, 'Quantum Lorentz Transformations,'

lorentzb.pdf

Allowed quantum numbers for Quarkonia

- Consider a $q\bar{q}$, where $q = \{u, d, s\}$, in a state of L and S
  $L =$ Orbital angular momentum $(= 0,\ 1,\ 2,\ 3, \cdots)^a$
  $S =$ Total intrinsic spin $(= 0,\ 1)^b$
- $P = (-)^{L+1}$ for any $q\bar{q}$ state[c]
- $C = (-)^{L+S}$ for a neutral $q\bar{q}$ state[c]
- $|L - S| \le J \le L + S$
- Forbidden J^{PC}'s: $(0^{--})^d$,
  0^{+-}, 1^{-+}, $(2^{+-})^e$, 3^{-+}, etc.

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[a] S. U. Chung,
"Spin Formalisms," CERN Yellow Report 71-8 (Updated)—spinfm1.pdf

[b] S. U. Chung,
"Quantum Lorentz Transformations"—lorentzb.pdf

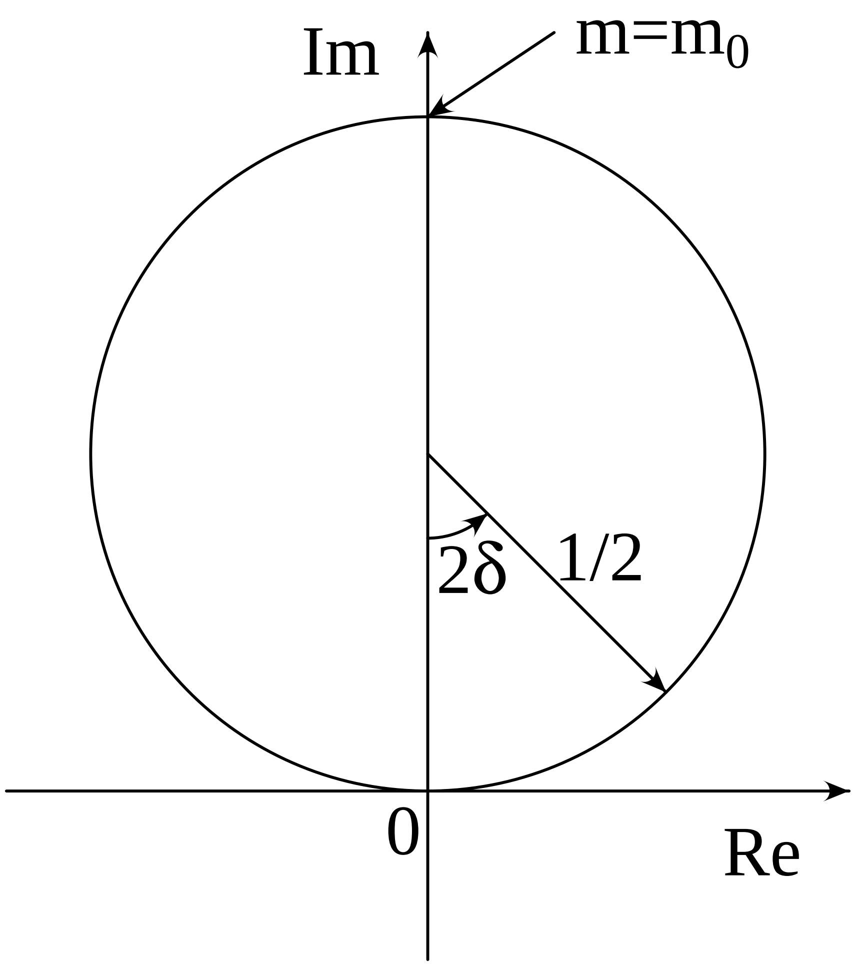
[c] S. U. Chung,
"*C*- and *G*-parity: a New Definition and Applications" (Version IV)—Cparity4.pdf

[d] S. U. Chung,
"Quantum Numbers for Hybrid mesons in the Flux-tube Model" (Version III)—flxtb0.pdf

[e] S. U. Chung,
"Photon-Pomeron Fusion Processes"

Phase motion of a Breit-Wigner form

Phase motion of a Breit-Wigner form

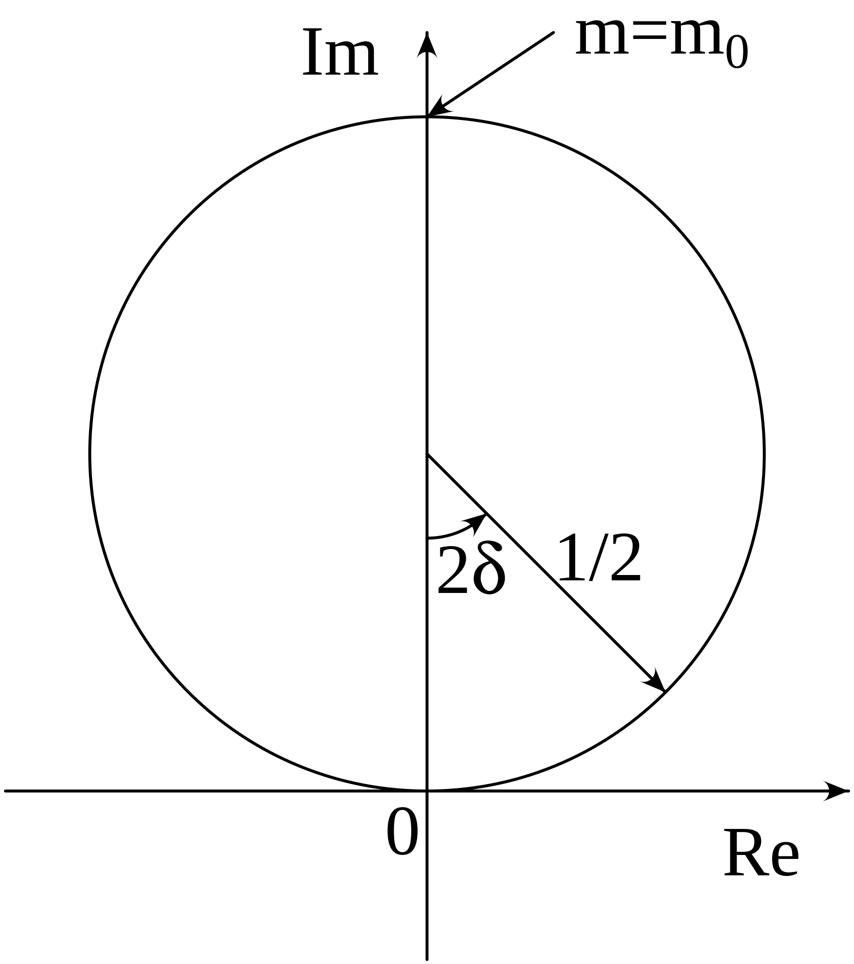


$m_0 =$ Mass (constant)
$\Gamma_0 =$ Width (constant)

$$\Delta(m) = \frac{m_0\,\Gamma_0}{m_0^2 - m^2 - i\,m_0\,\Gamma_0}$$
$$= \mathrm{e}^{i\,\delta(m)}\,\sin\delta(m)$$

$$\cot\delta(m) = \frac{m_0^2 - m^2}{m_0\,\Gamma_0}$$

Phase motion of a Breit-Wigner form



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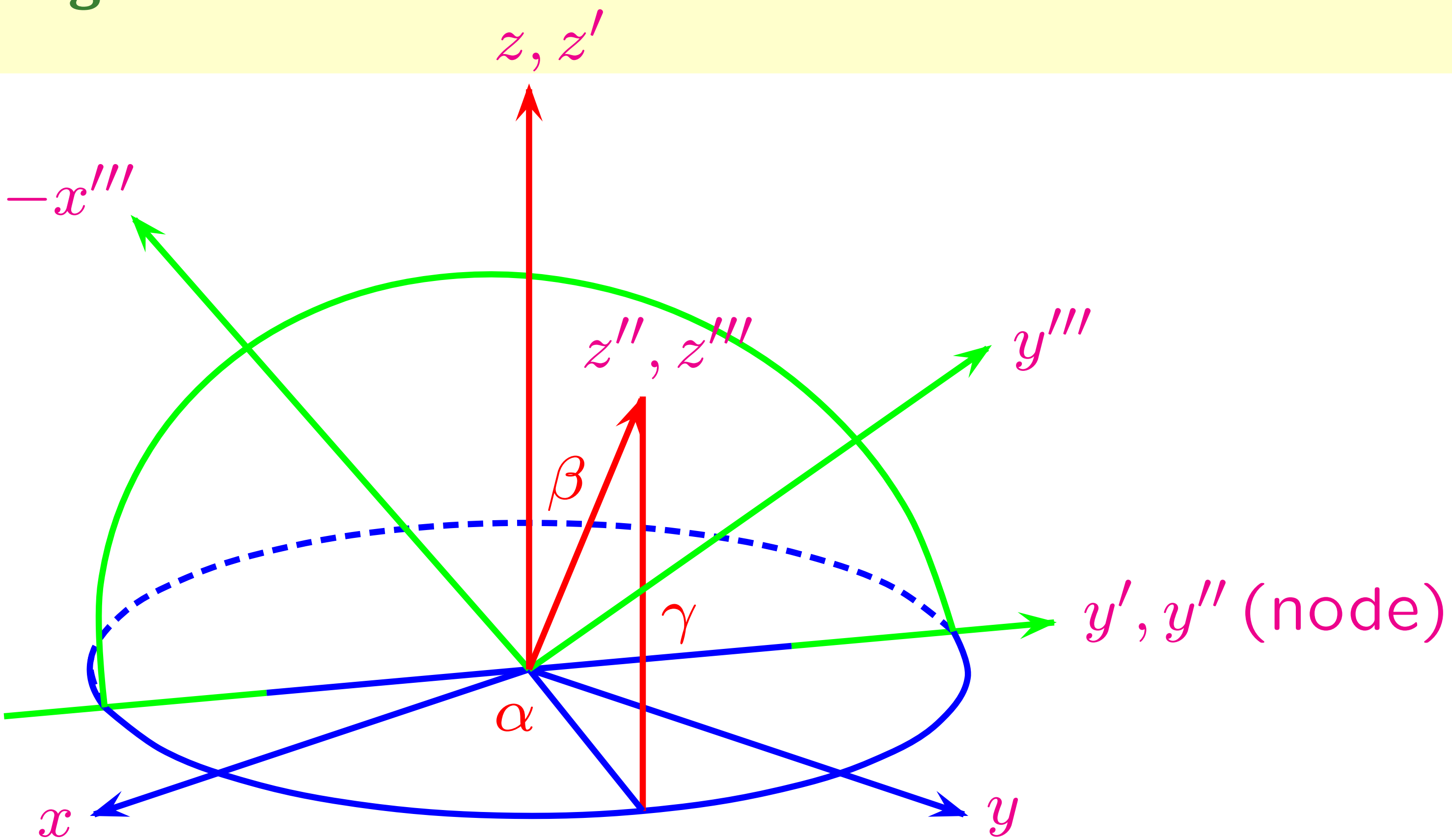
$\rho \to \pi\pi$:

$$\Delta(m) = \frac{m_0\,\Gamma_0}{m_0^2 - m^2 - i\,m_0\,\Gamma(m)}$$

$$\Gamma(m) = \Gamma_0\left(\frac{m_0}{m}\right)\left(\frac{q}{q_0}\right)^{(2\ell+1)}, \quad \ell = 1$$

$$\to \Gamma_0\left(\frac{m_0}{m}\right)\left(\frac{q}{q_0}\right)\left[\frac{F_\ell(q)}{F_\ell(q_0)}\right]^2$$

Euler Angles



$$U[R(\alpha,\beta,\gamma)] = \exp[-i\,\alpha\,J_z]\,\exp[-i\,\beta\,J_y]\,\exp[-i\,\gamma\,J_z]$$

$$= \exp[-i\,\gamma\,J''_z]\,\exp[-i\,\beta\,J'_y]\,\exp[-i\,\alpha\,J_z]$$

$$D^j_{m'm}(\alpha,\beta,\gamma) = \langle jm'|U[R(\alpha,\beta,\gamma)]|jm\rangle$$

M. E. Rose,
'Elementary Theory of Angular Momentum,'
John Wiley & Sons, Inc. (see Chapter IV)

General two-body decay for $J \longrightarrow s_1 + s_2$:

$$A_M^{J\lambda_1\lambda_2}(\Omega) = \langle \vec{p}\,\lambda_1; -\vec{p}\,\lambda_2 | \mathscr{M} | JM \rangle$$

$$= N_J\, F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0), \quad N_J = \sqrt{\frac{2J+1}{4\pi}}$$

where $\lambda = \lambda_1 - \lambda_2$. We have

$$F^J_{\lambda_1\lambda_2} = 4\pi \left(\frac{w}{p}\right)^{1/2} \langle JM\,\lambda_1\,\lambda_2 | \mathscr{M} | JM \rangle$$

$$= \sum_{\ell S} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} G^J_{\ell S}\,(\ell 0 S\lambda | J\lambda)\,(s_1\lambda_1 s_2 - \lambda_2 | S\lambda)$$

where

$$G^J_{\ell S} = 4\pi \left(\frac{w}{p}\right)^{1/2} \langle JM\ell S | \mathscr{M} | JM \rangle$$

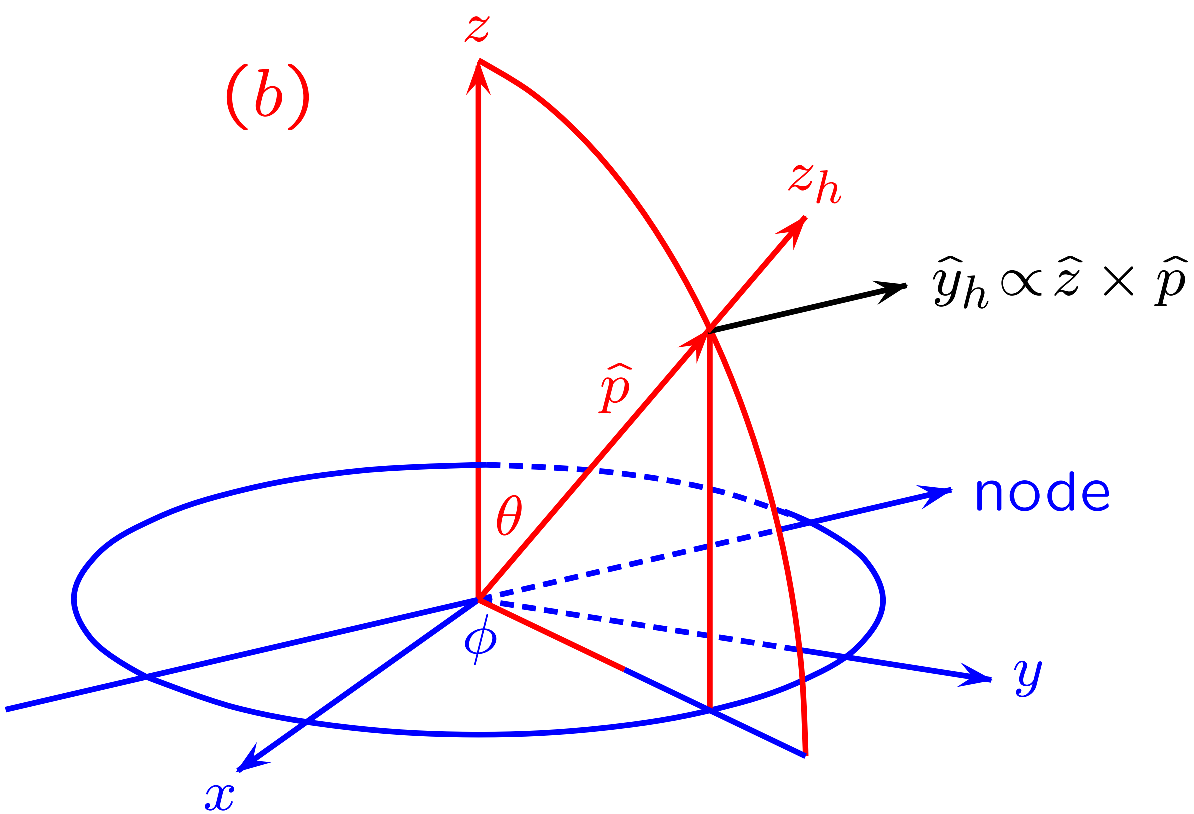
and

$$\sum_{\lambda_1\lambda_2} \left|F^J_{\lambda_1\lambda_2}\right|^2 = \sum_{\ell S} \left|G^J_{\ell S}\right|^2$$

Helicity states:

$$|\vec{p}, j\lambda\rangle = |\phi, \theta, p, j\lambda\rangle$$

$$= U[\mathring{R}(\phi, \theta, 0)]\, U[L_z(p)]\, |j\lambda\rangle = U[L(\vec{p}\,)]\, U[\mathring{R}(\phi, \theta, 0)]\, |j\lambda\rangle$$



Helicity coordinate system:

$$\hat{x}_h = \hat{y}_h \times \hat{z}_h \qquad \hat{y}_h \propto \hat{z} \times \hat{p}, \qquad \hat{z}_h = \hat{p}$$

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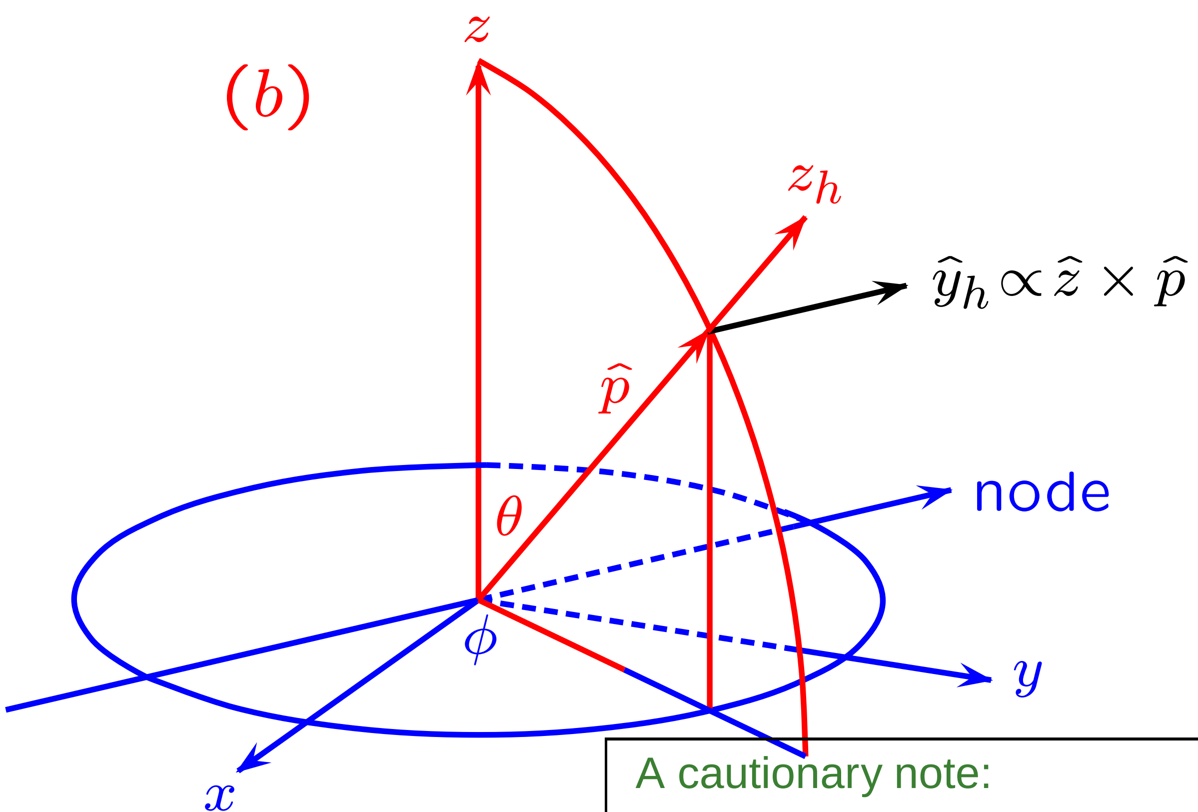

A cautionary note:
See a note by S. U. Chung
'Sequential Decays involving $\omega \to \gamma + \pi^0$'
omega6.pdf

Helicity coordinate system:

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Zemach Formalism:

C. Zemach, Nuovo Cimento 32, 1605 (1964)
C. Zemach, Phys. Rev. 140 B, 97 (1965)
S. U. Chung, 'Spin Formalisms,'
CERN 71-8 (Updated Version)—spinfm1.pdf

$$G^J_{\ell S} \implies G^J_{\ell S} \cdot p^\ell$$

where p is a breakup momentum in a suitable rest frame(RF).

BNL PWA Program:

$$G^J_{\ell S} \implies G^J_{\ell S} \cdot F_\ell(p);$$

$$F_\ell(p) \propto (p/p_R)^\ell, \quad \text{for} \quad p/p_R \to 0$$
$$F_\ell(p) \propto 1, \quad \text{for} \quad p/p_R \to \infty$$

where $F_\ell(p)$'s are the Blatt-Weisskopf barrier factors with $q_R = 0.1973$ GeV/c corresponding to 1.0 fermi.

Blatt-Weisskopf centrifugal-barrier factors

F. von Hippel and C. Quigg, Phys. Rev. **5**, 624 (1972)

S. U. Chung, 'Formulas for Angular-Momentum Barrier Factors,'

brfactor1.pdf

The radial Schrödinger equation with the potential $V(r) = 0$ for $r > R$ ("interaction radius") is

$$\left[\frac{d^2}{d\rho^2} + 1 - \frac{\ell(\ell+1)}{\rho^2}\right] U_\ell(\rho) = 0, \quad U_\ell(\rho) = \rho\, h_\ell^{(1)}(\rho)$$

where $\rho = p\,r$. The outgoing-wave solutions are given by $U_\ell(\rho)$ with

$$h_\ell^{(1)}(\rho) = \text{Sherical Hankel functions of the first kind}$$

$$U_\ell(\rho) = \rho\, h_\ell^{(1)}(\rho) = -i \exp\left[i\left(\rho - \frac{\pi}{2}\,\ell\right)\right] \sum_{k=0}^{\ell} (-1)^k \frac{(\ell+k)!}{k!\,(\ell-k)!}\,(2i\rho)^{-k}$$

Note

$$\left|U_\ell(\rho)\right|_{\rho\to 0} \propto \rho^{-\ell}, \qquad \left|U_\ell(\rho)\right|_{\rho\to\infty} = 1$$

Radial Wave Functions

$\rho(770) \to \pi\pi$

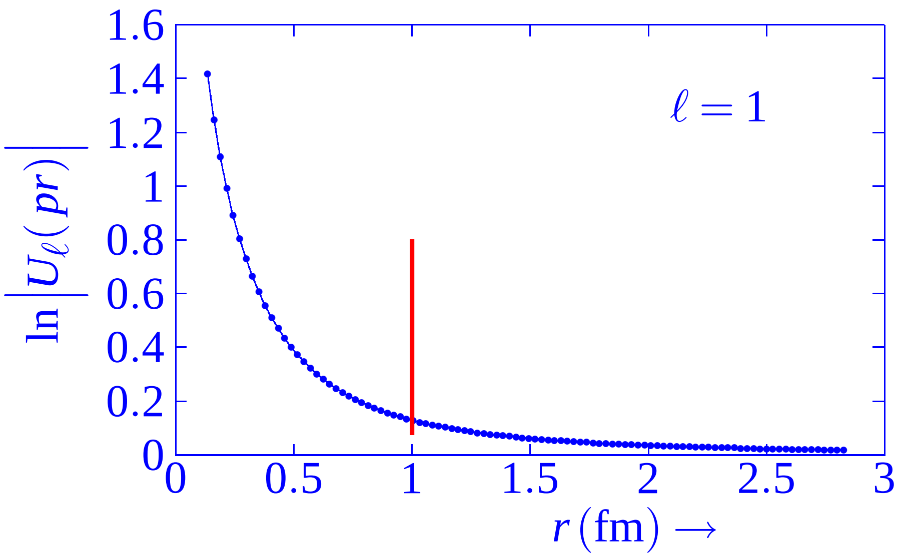
$p = 0.363$ GeV

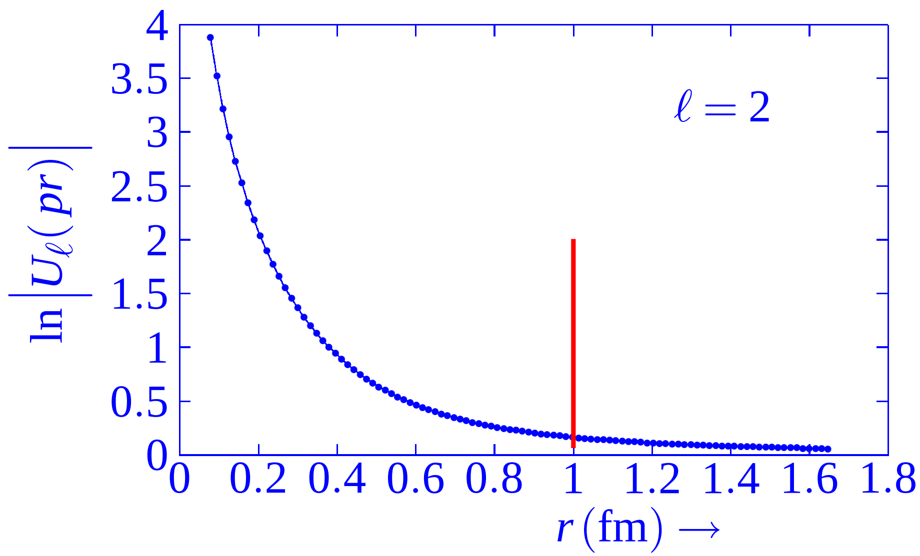
$r_p = \hbar c/p = 0.544$ fm

$f_2(1270) \to \pi\pi$

$p = 0.623$ GeV

$r_p = \hbar c/p = 0.317$ fm





$r = R = 1$ fm;

$p_R = 0.1973$ GeV;

$\ln\left|U_1(pR)\right| = 0.1282$

$\left|U_1(p/p_R)\right| = 1.137$

$\left|U_1(1.840)\right| = 1.137$

$r = R = 1$ fm;

$p_R = 0.1973$ GeV;

$\ln\left|U_2(pR)\right| = 0.1606$

$\left|U_2(p/p_R)\right| = 1.174$

$\left|U_2(3.158)\right| = 1.174$

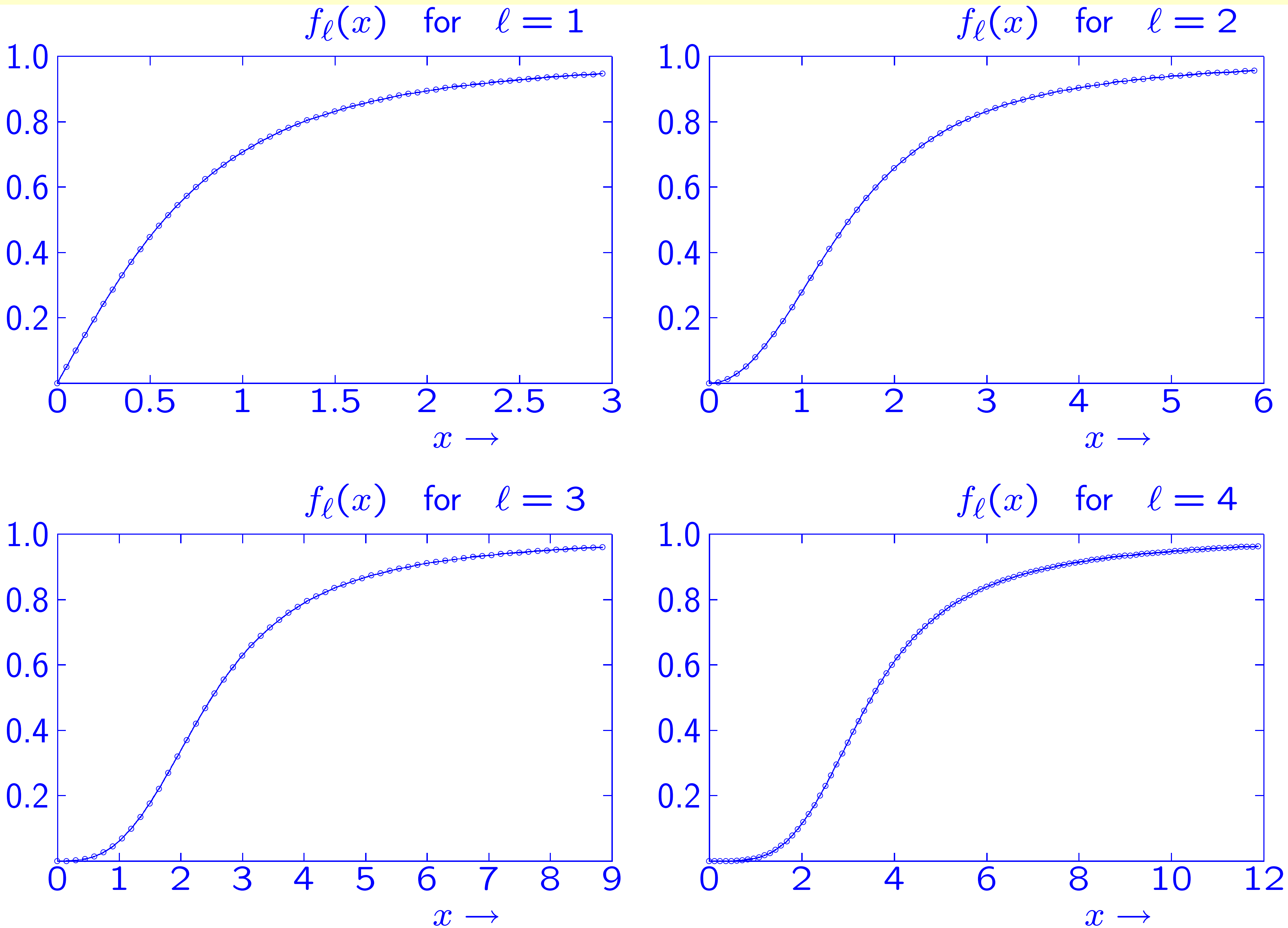
Define Blatt-Weisskopf barrier factors via

$$f_\ell(x) = \frac{\left|U_\ell(pr)\right|_{r\to\infty}}{\left|U_\ell(pr)\right|_{r=R}} = \frac{1}{x\left|h_\ell^{(1)}(x)\right|}, \quad x = pR = p/p_R$$

where $R = 1$ fm and $p_R = 0.1973$ GeV/c. Observe

$$f_0(x) = 1, \quad f_1(x) = \frac{x}{\sqrt{x^2+1}}, \quad f_2(x) = \frac{x^2}{\sqrt{(x^2-3)^2+9x^2}} \cdots$$

Barrier Factors: Examples



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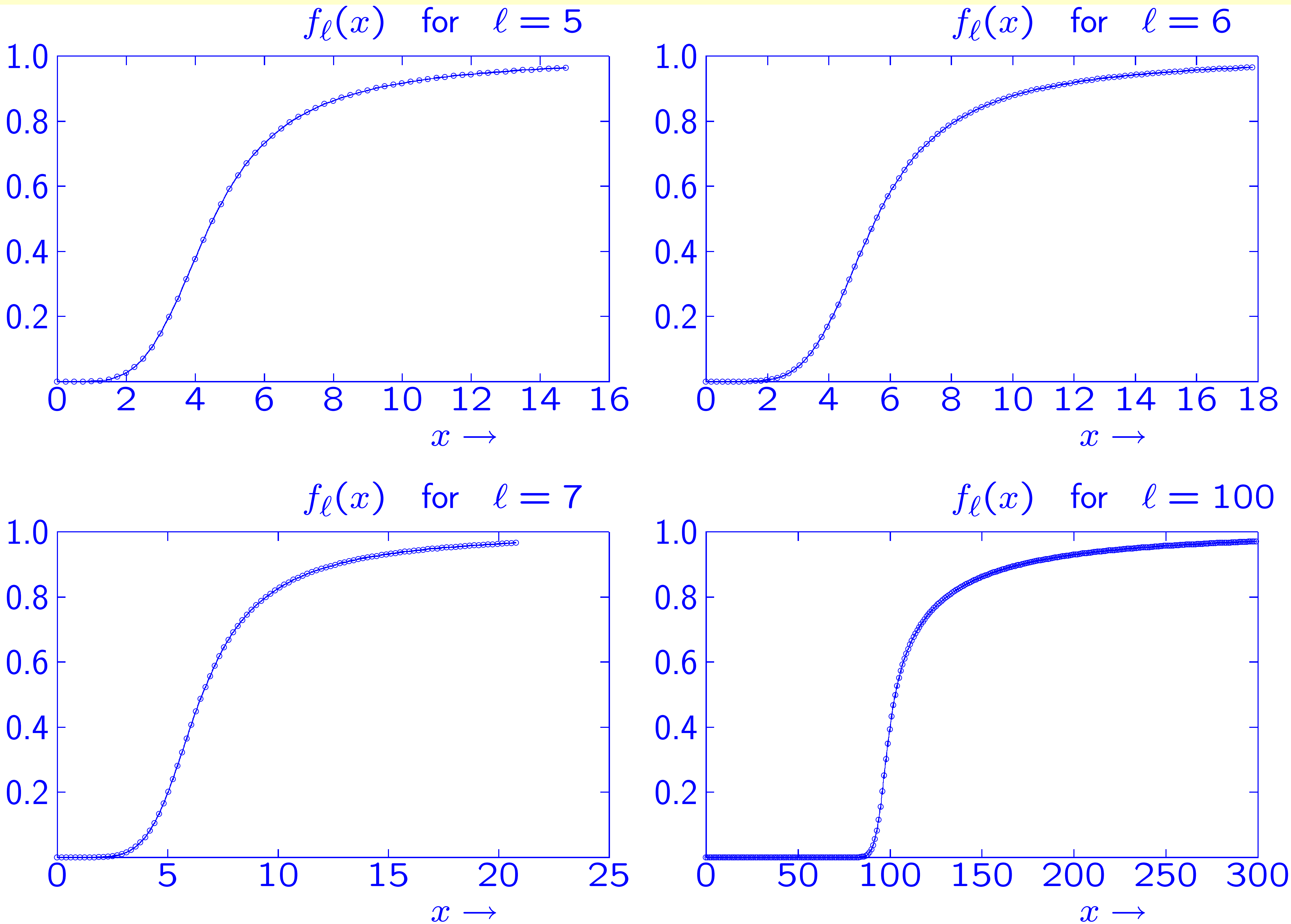
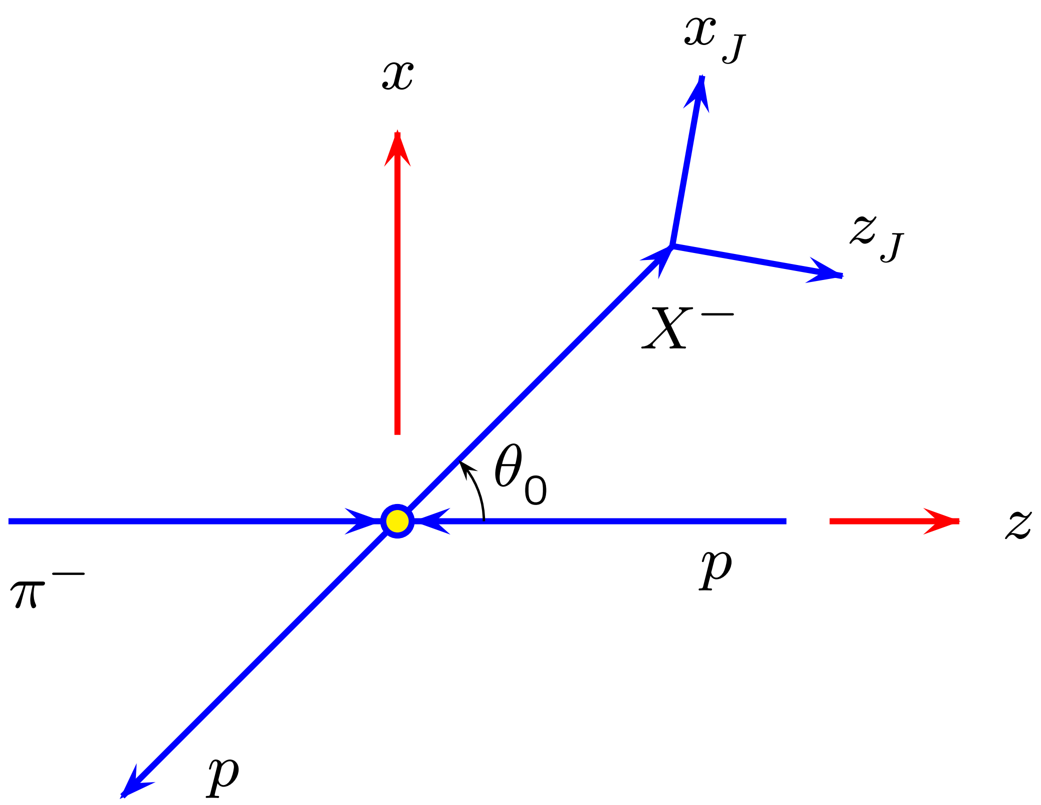
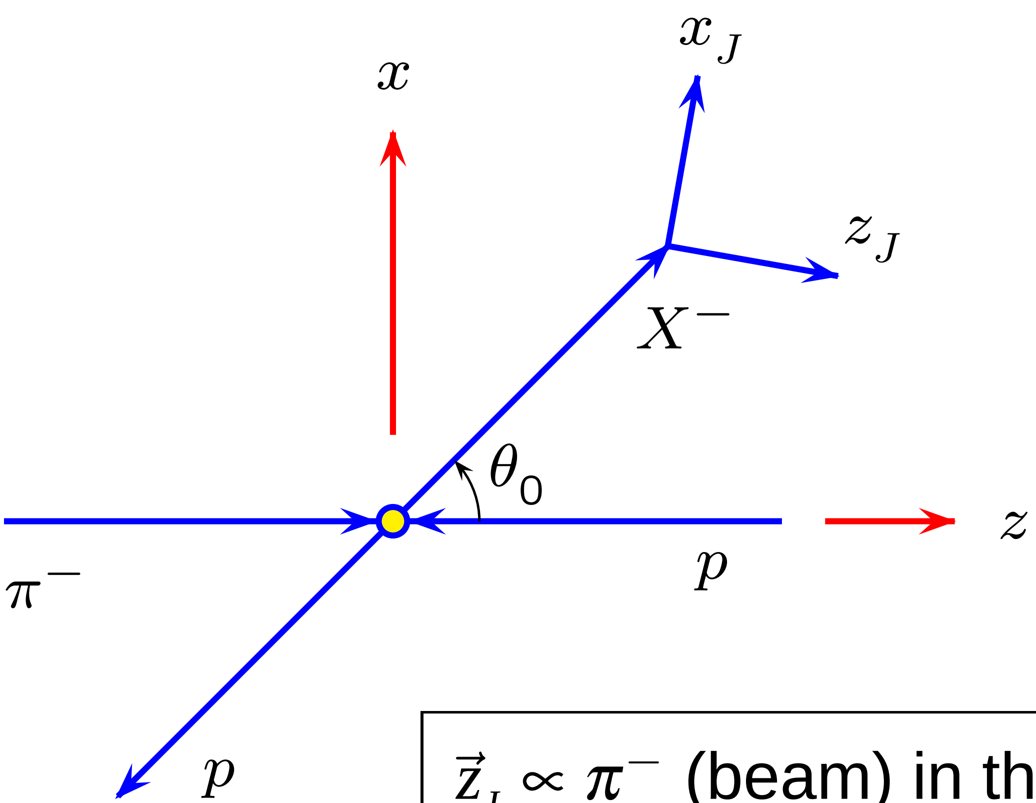


$a(\pi^-) + b(p) \to c(X^-) + d(p)$:



$$a(\pi^-) + b(p) \rightarrow c(X^-) + d(p):$$



$\vec{z}_J \propto \pi^-$ (beam) in the X rest frame (XRF)

$$t' = |t| - |t|_{\min}$$

$$= 2\, p_i\, p_f\, (1 - \cos\theta_0) \geq 0$$

Reflection Operator: $\Pi_y(\pi)$

General Angular Distributions in Reflectivity Basis:

S. U. Chung and T. L. Trueman, Phys. Rev. **D11**, 633 (1975)

Consider

$$a(\pi^-) + b(p) \to c(X^-) + d(p), \quad X^-(\chi) \to \pi^+\pi^-\pi^-,\ K^0K^\mp\pi^\pm$$

Let $|jm\rangle$ be the spin state for c where the quantization axis is defined in the production plane, i.e. one takes either the helicity or the Jackson frame for c. The amplitude for production and decay of the c is

$$A \propto \sum_{\chi m} \langle \vec{p}_c\, \chi m,\ \vec{p}_d \lambda_d \,|T|\, \vec{p}_a \lambda_a,\ \vec{p}_b \lambda_b \rangle D^{\chi}_{m}(\tau)$$

where χ specifies the complete quantum state for c, which includes its spin j, its parity, its C-parity, its isotopic spin, and its decay products with the phase-space element given by τ; λ's refer to the helicities; T is the transition operator of the process $ab \to cd$; and D is the decay amplitude for c, which may consist of a product of the 'rotation functions' as well as the Breit-Wigner forms.

The distribution function follows immediately

$$I(\tau) \propto \sum_{\lambda_a \lambda_b \lambda_d} \sum_{\chi m\, \chi' m'} \times \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c\, \chi' m',\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^* \times D^{\chi}_{m}(\tau)\, D^{\chi'\,*}_{m'}(\tau)$$

Note that the helicities of a, b and d are the 'external' unobserved variables and therefore summed over outside of the absolute square of the amplitude A. In terms of the generalized spin-density matrix

$$\rho^{\chi\chi'}_{mm'} \propto \sum_{\lambda_a \lambda_b \lambda_d} \langle \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c\, \chi' m',\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^*$$

the distribution function assumes an elegant form

$$I(\tau) \propto \sum_{\substack{\chi m \\ \chi' m'}} \rho^{\chi\chi'}_{mm'}\, D^{\chi}_{m}(\tau)\, D^{\chi'\,*}_{m'}(\tau)$$

We are now ready to introduce the reflection operator through the production plane for $ab \to cd$. Let this plane be defined to be the x-z plane, i.e. the production normal is along the y-axis. Then the reflection operator defined by

$$\Pi_y = U[R_y(\pi)]\,\Pi = \Pi\,U[R_y(\pi)]$$

where $U[R_y(\pi)]$ is a unitary operator representing a rotation by π around the y-axis, i.e.

$$U[R_y(\pi)] = \exp(-i\pi\,J_y)$$

which is simply given by the standard d-function

$$U_{m'\,m}[R_y(\pi)] = d^{\,j}_{m'\,m}(\pi) = (-)^{j-m}\,\delta_{m',\,-m}$$

Note also that

$$U_{m'\,m}[R_y(-\pi)] = d^{\,j}_{m'\,m}(-\pi) = (-)^{j+m}\,\delta_{m',\,-m}$$

Let Λ is a general operator in the xz-plane. Then, we see that

$$\left[\Pi_y, U[\Lambda(\vec{p}_c)]\right] = 0, \quad \Pi_y\,|\vec{p}_c\,\chi\,m\rangle = \eta_c\,(-)^{j-m}\,|\vec{p}_c\,\chi\,-m\rangle, \qquad \Pi_y^2 = (-)^{2j}\,I$$

where η_c is the intrinsic parity of the c. Also true for helicity states ($|\vec{p}_i\,\lambda_i\rangle$, $i = a, b, d$). Also true for massless particles ($m \to \lambda = \pm j$).

We move over to the reflection-basis states for c, i.e.

$$|\vec{p}_c\,\varepsilon\,\chi\,m\rangle = \theta(m)\left\{|\vec{p}_c\,\chi\,m\rangle + \varepsilon\,\eta_c\,(-)^{j-m}\,|\vec{p}_c\,\chi\,-m\rangle\right\}, \quad \eta_c = \text{intrinsic parity of } c$$

where

$$\theta(m) = \frac{1}{\sqrt{2}},\ m>0; \qquad \theta(m) = \frac{1}{2},\ m=0; \qquad \theta(m) = 0,\ m<0$$

These basis states constitute eigenvectors of the reflection operator

$$\varepsilon^2 = (-)^{2j} \quad \rightarrow \quad \Pi_y\,|\vec{p}_c\,\varepsilon\,\chi\,m\rangle = \varepsilon\,(-)^{2j}\,|\vec{p}_c\,\varepsilon\,\chi\,m\rangle$$

so that we have $\varepsilon = \pm 1$ for bosons and $\varepsilon = \pm i$ for fermions. Note, in addition, that $\varepsilon\varepsilon^* = |\varepsilon|^2 = 1$ for both bosons and fermions.
The generalized density matrix in the reflectivity basis is, with $m \geq 0$ and $m' \geq 0$,

$$^{\varepsilon\,\varepsilon'}\rho^{\chi\,\chi'}_{mm'} \propto \sum_{\lambda_a\lambda_b\lambda_d} \langle\varepsilon\,\vec{p}_c\,\chi m,\,\vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\,\vec{p}_b\lambda_b\rangle\langle\varepsilon'\,\vec{p}_c\,\chi' m',\,\vec{p}_d\lambda_d\,|T|\,\vec{p}_a\lambda_a,\,\vec{p}_b\lambda_b\rangle^*$$

We shall explore the consequences of a reflection operation applied to the transition matrices above. The key observation is that Π_y leaves the transition operator T unperturbed, i.e. $[\Pi_y, T] = 0$. The three-vectors $\vec{p}_i$ $(i = a, b, c, d)$ are left unchanged under Π_y, i.e. the boost operators and/or the rotations about the y-axis, which enter in the definitions of the helicity or the Jackson frames, remain invariant under Π_y. Therefore, the parity conservation is relegated to exploring the consequences of Π_y acting on the 'rest' states. Insert $\Pi_y^{-1}\,\Pi_y = \Pi_y^{\dagger}\,\Pi_y = I$ next to each T and propagate $\Pi_y^{\dagger}$ and Π_y backwards and forwards, respectively, to find

$$
{}^{\varepsilon\,\varepsilon'}\rho^{\chi\,\chi'}_{mm'} \propto \varepsilon\varepsilon'^{\,*}\,(-)^{2(j-j')} \sum_{\lambda_a\lambda_b\lambda_d}
$$
$$
\langle\varepsilon\,\vec{p}_c\,\chi m,\,\vec{p}_d\,,-\lambda_d\,|T|\,\vec{p}_a\,,-\lambda_a,\,\vec{p}_b\,,-\lambda_b\rangle\,\langle\varepsilon'\,\vec{p}_c\,\chi' m',\,\vec{p}_d\,,-\lambda_d\,|T|\,\vec{p}_a\,,-\lambda_a,\,\vec{p}_b\,,-\lambda_b\rangle^{*}
$$

so that

$$
{}^{\varepsilon\,\varepsilon'}\rho^{\chi\,\chi'}_{mm'} = \varepsilon\,\varepsilon'^{\,*} \times {}^{\varepsilon\,\varepsilon'}\rho^{\chi\,\chi'}_{mm'}
$$

So we see that $\varepsilon\,\varepsilon'^{\,*} = +1$. Multiply it by ε' from the right and noting that $\varepsilon'\,\varepsilon'^{\,*} = |\varepsilon'|^2 = +1$, we find $\varepsilon = \varepsilon'$. Here we have carefully handled the derivation, so that the formula above applies to both bosons and fermions.

We shall explore the consequences of a reflection operation applied to the transition matrices above. The key observation is that Π_y leaves the transition operator T unperturbed, i.e. $[\Pi_y, T] = 0$. The three-vectors $\vec{p}_i$ $(i = a, b, c, d)$ are left unchanged under Π_y, i.e. the boost operators and/or the rotations about the y-axis, which enter in the definitions of the helicity or the Jackson frames, remain invariant under Π_y. Therefore, the parity conservation is relegated to exploring the consequences of Π_y acting on the 'rest' states. Insert ${\Pi_y}^{-1}\,\Pi_y = {\Pi_y}^{\dagger}\,\Pi_y = I$ next to each T and propagate $\Pi_y^{\dagger}$ and Π_y backwards and forwards, respectively, to find

$$ {}^{\varepsilon\varepsilon'}\rho^{\chi\chi'}_{mm'} \propto \varepsilon\varepsilon'^{\,*}\,(-)^{2(j-j')} \sum_{\lambda_a\lambda_b\lambda_d} $$

$$ \langle \varepsilon\,\vec{p}_c\,\chi m,\, \vec{p}_d\,, -\lambda_d\,|T|\,\vec{p}_a\,, -\lambda_a,\, \vec{p}_b\,, -\lambda_b\rangle\,\langle \varepsilon'\,\vec{p}_c\,\chi' m',\, \vec{p}_d\,, -\lambda_d\,|T|\,\vec{p}_a\,, -\lambda_a,\, \vec{p}_b\,, -\lambda_b\rangle^{*} $$

so that

$$ {}^{\varepsilon\varepsilon'}\rho^{\chi\chi'}_{mm'} = \varepsilon\,\varepsilon'^{\,*} \times {}^{\varepsilon\varepsilon'}\rho^{\chi\chi'}_{mm'} $$

So we see that $\varepsilon\,\varepsilon'^{\,*} = +1$. Multiply it by ε' from the right and noting that $\varepsilon'\,\varepsilon'^{\,*} = |\varepsilon'|^2 = +1$, we find $\varepsilon = \varepsilon'$. Here we have carefully handled the derivation, so that the formula above applies to both bosons and fermions.

S. U. Chung,
'General Spin-Density Matrix'
spinmtrx1.pdf

The density matrix can be written, quite generally, in a block-diagonal form

$$
{}^{\varepsilon}\rho^{\chi\chi'}_{mm'} \propto \sum_{\lambda_a\lambda_b\lambda_d} \langle \varepsilon\, \vec{p}_c\, \chi m,\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle \langle \varepsilon\, \vec{p}_c\, \chi' m',\, \vec{p}_d \lambda_d\, |T|\, \vec{p}_a \lambda_a,\, \vec{p}_b \lambda_b \rangle^{*}
$$

a fundamental formula which incorporates parity conservation in the production process $ab \to cd$. The distribution function in the reflectivity basis is

$$
I(\tau) \propto \sum_{\varepsilon}^{2} \sum_{\substack{\chi m \\ \chi' m'}} {}^{\varepsilon}\rho^{\chi\chi'}_{mm'}\; {}^{\varepsilon}D^{\chi}_{m}(\tau)\; {}^{\varepsilon}D^{\chi'\,*}_{m'}(\tau), \qquad m \geq 0,\; m' \geq 0
$$

where ${}^{\varepsilon}D$ is the decay amplitude in the reflectivity basis. Consider a simple decay

$$
c \longrightarrow s_1(\lambda_1) + s_2(\lambda_2)
$$

In the cRF, we have, with $\vec{p}_c = 0$ and $\tau = R(\phi, \theta, 0)$,

$$
\begin{aligned}
{}^{\varepsilon}D^{\chi}_{m}(\tau) &= \langle \vec{q}\lambda_1; -\vec{q}\lambda_2 | \mathscr{M} | \varepsilon\, jm \rangle = \langle \vec{q}\lambda_1; -\vec{q}\lambda_2 | \lambda_1 \lambda_2\, \varepsilon\, jm \rangle \langle \lambda_1 \lambda_2\, \varepsilon\, jm | \mathscr{M} | \varepsilon\, jm \rangle \\
&= N_j\, F^{j}_{\lambda_1\, \lambda_2}\;\; \theta(m) \left\{ D^{j\,*}_{m\lambda}(R) + \varepsilon\, \eta_c\, (-)^{j-m}\, D^{j\,*}_{-m\lambda}(R) \right\}, \quad \lambda = \lambda_1 - \lambda_2
\end{aligned}
$$

An Example: $X \to \rho + \pi, \quad \rho \to \pi + \pi$

Let $s = 1$ and λ be the ρ spin and helicity. Define

$$W = \text{Effective mass of the } 3\pi \text{ system}$$

$$w = \text{Effective mass for } \rho \to \pi\pi$$

$$p = \text{Breakup momentum between w and the bachelor } \pi \text{ in } X\text{RF}$$

$$(\Theta, \Phi) = \text{The orientation of } \vec{p} \text{ in } X\text{RF}$$

$$q = \text{Breakup momentum for } \rho \to \pi\pi \text{ in } \rho\text{RF}$$

$$(\theta, \phi) = \text{The orientation of } \vec{q} \text{ in } \rho\text{RF}$$

Then, the full decay amplitude is

$$^{\varepsilon}D^{\chi}_{m}(\tau) = \sum_{\lambda} {}^{\varepsilon}D^{j\,\eta_c}_{m}(\tau)\, F_{\ell}(p)\, \Delta(w)\, F_s(q)\, D^{s\,*}_{\lambda\,0}(\phi, \theta, 0)$$

$$^{\varepsilon}D^{j\,\eta_c}_{m}(\tau) = N_j\, F^{j}_{\lambda}\; \theta(m) \left\{ D^{j\,*}_{m\lambda}(\Phi, \Theta, 0) + \varepsilon\, \eta_c\, (-)^{j-m}\, D^{j\,*}_{-m\lambda}(\Phi, \Theta, 0) \right\}$$

$$F^{j}_{\lambda} = \left(\frac{2\ell + 1}{2j + 1} \right)^{\frac{1}{2}} (\ell 0 s \lambda | J \lambda)$$

Note the decay amplitude $G^{j}_{\ell s}$ has been absorbed into the production amplitudes (the variables to be fitted experimentally).

The rank of the density matrix is determined by the number of independent terms in the summation on helicities. Let

$$n_i = 2s_i + 1 \quad \text{or} \quad n_i = 2\,(\text{photons}) \quad \text{for} \quad i = a,\ b,\ d$$

depending on whether a particle is massive or massless. The total number in the sum is

$$N = n_a\, n_b\, n_d$$

So the rank of the density matrix is $(N+1)/2$ if N is odd, and it is $N/2$ if N is even. Note that the reduction in the rank comes from parity conservation in the production process (to show this, apply Π_y again to the amplitudes).

Table I. Rank of Spin-Density Matrix for X

Reaction			Rank
$\pi^- p$	$\longrightarrow$	$\pi^- X^+$	1
$\pi^- p$	$\longrightarrow$	$X^- p$	2
$\pi^+ n$	$\longrightarrow$	$X^- \Delta^{++}$	4
$\bar{p}p$	$\longrightarrow$	$X^- p$	4
$\bar{n}p$	$\longrightarrow$	$X^- \Delta^{++}$	8
γp	$\longrightarrow$	$X^0 p$	4
γp	$\longrightarrow$	$\pi^+ X^0$	2
νp	$\longrightarrow$	$e^- X^{++}$	$1^\dagger$
$e^- p$	$\longrightarrow$	$e^- X^+$	$1^\dagger$
ϕp	$\longrightarrow$	$X^0 p$	6
ϕp	$\longrightarrow$	$\pi^+ X^0$	3
$\pi^- \eta$	$\longrightarrow$	$\pi^- X^0$	1
$\pi^- \phi$	$\longrightarrow$	$\pi^- X^0$	2

† The electrons are assumed to come with one helicity.
True in general for odd-half-integer spins in the limit of zero mass.

Consider an $N_\varepsilon \times N_\varepsilon$ density matrix, with $i = \{\chi\, m\}$ and $j = \{\chi'\, m'\}$,

$$ {}^\varepsilon\rho_{ij} = \sum_{k=1}^{K_\varepsilon} {}^\varepsilon V_{ik}\, {}^\varepsilon V^*_{jk}, \quad \Longrightarrow \quad {}^\varepsilon\rho = {}^\varepsilon V\, {}^\varepsilon V^\dagger, \quad \Longrightarrow \quad {}^\varepsilon\rho = {}^\varepsilon\rho^\dagger $$

where $i, j = 1, \cdots, N_\varepsilon$; $k = 1, \cdots, K_\varepsilon$, and $K_\varepsilon\, (= 1, \cdots, \infty)$ is the rank of the density matrix. Note that ${}^\varepsilon\rho$ is an $N_\varepsilon \times N_\varepsilon$ square matrix, whereas ${}^\varepsilon V$ is, in general, a retangular matrix $N_\varepsilon \times K_\varepsilon$. The 'Cholesky' decomposition of ${}^\varepsilon V$ is, e.g. for $N_\varepsilon = 6$ or for $6 \times 6\ {}^\varepsilon\rho$,

$$ {}^\varepsilon V = \{{}^\varepsilon V_{ik}\} = \begin{pmatrix} {}^\varepsilon V_{11} & 0 & 0 & 0 & 0 & 0 \\ {}^\varepsilon V_{21} & {}^\varepsilon V_{22} & 0 & 0 & 0 & 0 \\ {}^\varepsilon V_{31} & {}^\varepsilon V_{32} & {}^\varepsilon V_{33} & 0 & 0 & 0 \\ {}^\varepsilon V_{41} & {}^\varepsilon V_{42} & {}^\varepsilon V_{43} & {}^\varepsilon V_{44} & 0 & 0 \\ {}^\varepsilon V_{51} & {}^\varepsilon V_{52} & {}^\varepsilon V_{53} & {}^\varepsilon V_{54} & {}^\varepsilon V_{55} & 0 \\ {}^\varepsilon V_{61} & {}^\varepsilon V_{62} & {}^\varepsilon V_{63} & {}^\varepsilon V_{64} & {}^\varepsilon V_{65} & {}^\varepsilon V_{66} \end{pmatrix} $$

where ${}^\varepsilon V_{ik}$ is complex in general but ${}^\varepsilon V_{ii} =$ real ≥ 0. There are 6 real diagonal elements and 15 complex off-diagonal elements of ${}^\varepsilon V$, for a total of 36 parameters required to describe a $6 \times 6\ {}^\varepsilon\rho$. The rank is given by the number of columns counting from the left, with the rest being zero. For example, if the rank=2, then we must have ${}^\varepsilon V_{ik} = 0$, $i \geq k \geq 3$. In this case, there are 2 real diagonal elements and 9 complex off-diagonal elements, for a total of 20 parameters in the problem.

Now go back to the notation $\{\chi\, m\}$ and $\{\chi'\, m'\}$

$$ {}^{\varepsilon}\rho^{\chi\,\chi'}_{mm'} = \sum_{k=1}^{K_\varepsilon} {}^{\varepsilon}V^{\chi}_{m\,k}\; {}^{\varepsilon}V^{\chi'\,*}_{m'\,k} \quad , \qquad m \geq 0,\; m' \geq 0 $$

and define

$$ {}^{\varepsilon}U_k(\tau) = \sum_{\chi\, m}^{N_\varepsilon} {}^{\varepsilon}V^{\chi}_{m\,k}\; {}^{\varepsilon}D^{\chi}_{m}(\tau), \quad m \geq 0 $$

The distribution function

$$ I(\tau) \propto \sum_{\varepsilon}^{2} \sum_{\substack{\chi m \\ \chi' m'}}^{N_\varepsilon} {}^{\varepsilon}\rho^{\chi\,\chi'}_{mm'}\; {}^{\varepsilon}D^{\chi}_{m}(\tau)\; {}^{\varepsilon}D^{\chi'\,*}_{m'}(\tau) $$

becomes

$$ I(\tau) \propto \sum_{\varepsilon}^{2} \sum_{k=1}^{K_\varepsilon} \left| {}^{\varepsilon}U_k(\tau) \right|^2 $$

Experimental Moments

Let $R(\alpha,\beta,\gamma)$ describe the orientation of the 3π system in the XRF. Compare the unnormalized experimental moments

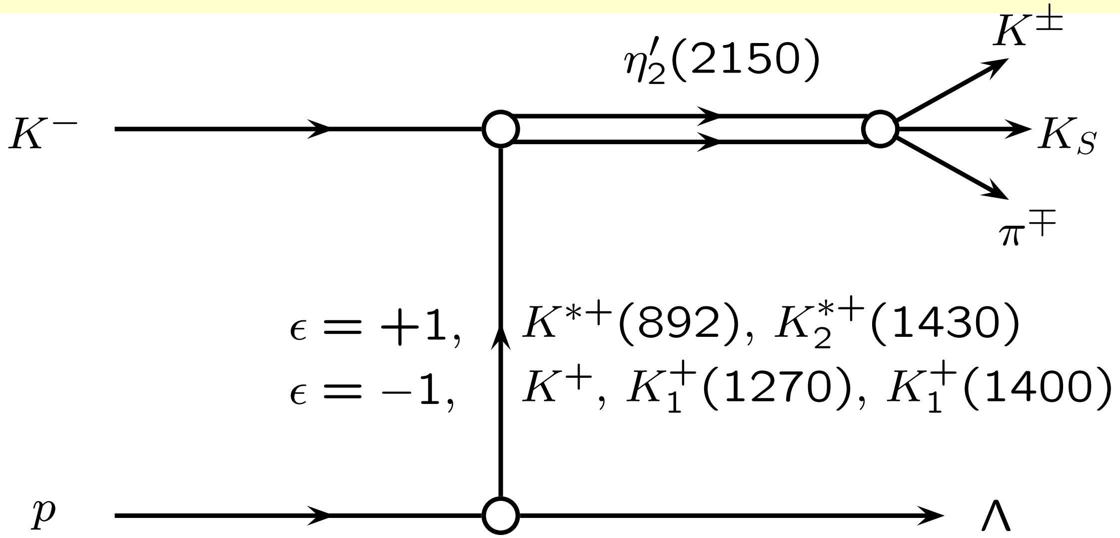
$$H_x(LMN) = \sum_i^n D^L_{MN}(R_i)$$

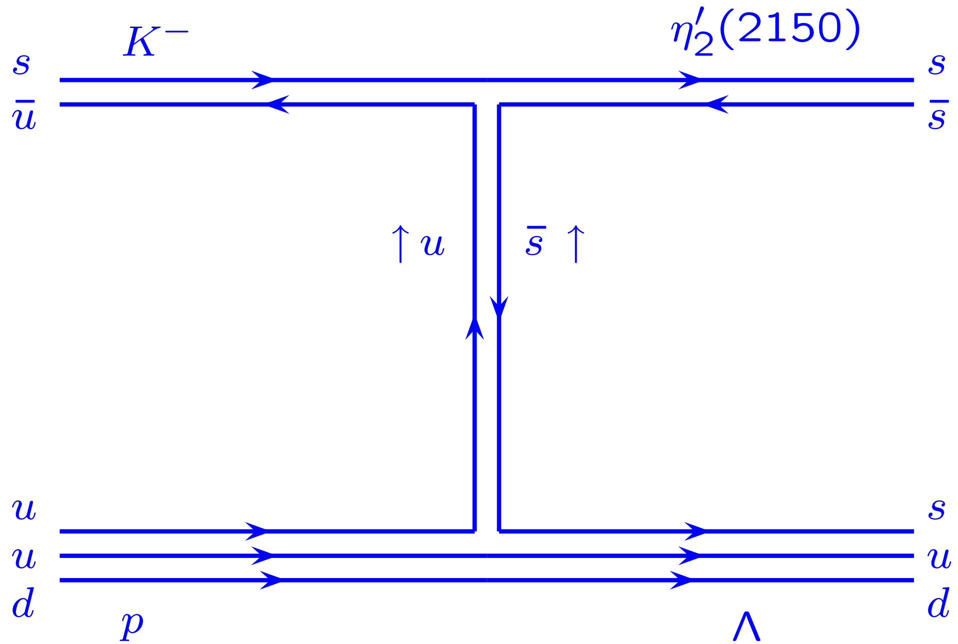
with the predicted ones

$$H_x(LMN) = \int \eta(R)\, I(R)\, D^L_{MN}(R)\, dR$$

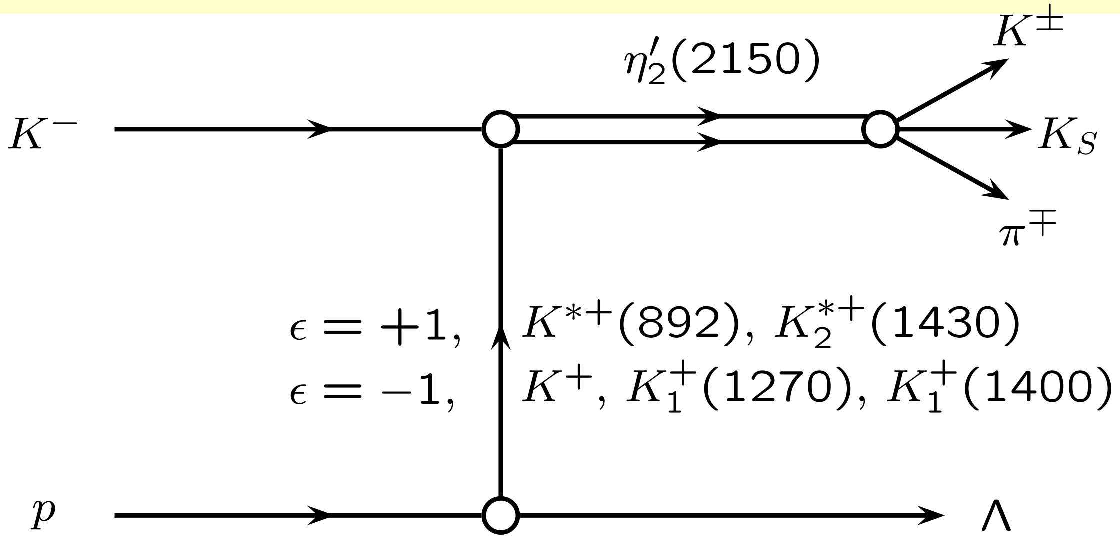
The body-fixed z axis is chosen as the normal to the decay plane. This is done for each region of the Dalitz plot.

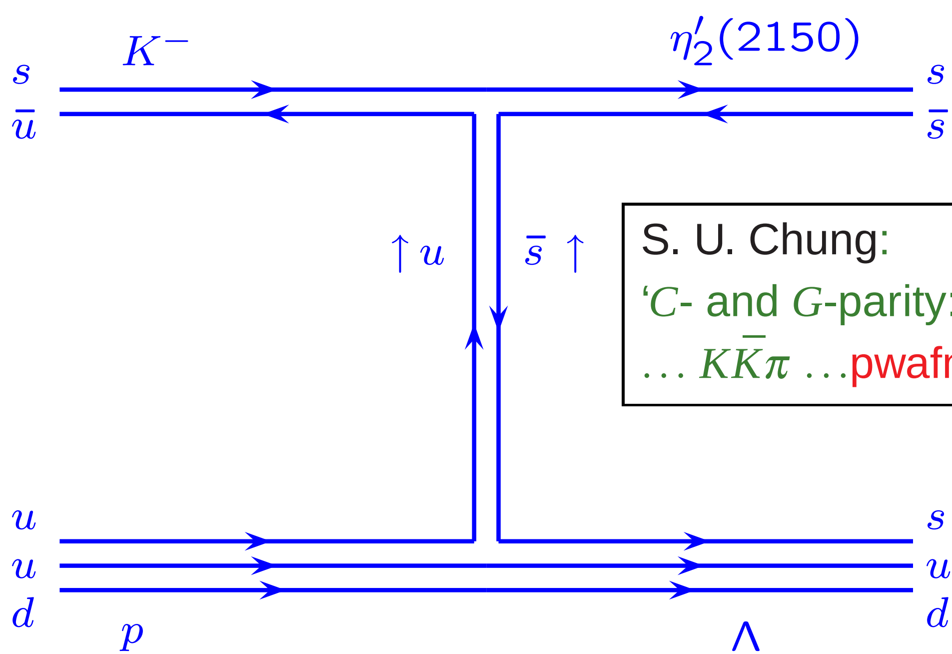
Strangeonium Excitations





Strangeonium Excitations





S. U. Chung:
'C- and G-parity: ...' Cparity4.pdf
... $K\bar{K}\pi$ ...pwafm0.pdf, kkbarpi1.pdf

Maximum-Likelihood Method

Introduce the so-called extended likelihood function for finding ‘n’ events in a given mass bin

$$\mathscr{L} \propto \left[\frac{\bar{n}^n}{n!}\,\mathrm{e}^{-\bar{n}}\right] \prod_i^n \left[\frac{I(\tau_i)}{\int I(\tau)\,\eta(\tau)\,\phi(\tau)\,\mathrm{d}\tau}\right]$$

no phase-space factors in $I(\tau_i)$

where $\eta(\tau)$ is the experimental finite acceptance at τ and the invariant phase-space element given by

$$\mathrm{d}\phi = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)\mathrm{d}\tau = \phi(\tau)\,\mathrm{d}\tau$$

The first bracket in $\mathscr{L}$ represents the Poisson probability for finding ‘n’ events in the mass bin, and the expectation value $\bar{n}$ is

$$\bar{n} \propto \int I(\tau)\,\eta(\tau)\,\phi(\tau)\,\mathrm{d}\tau$$

The likelihood function $\mathscr{L}$ can now be written, dropping the factors depending on n alone,

$$\mathscr{L} \propto \left[\prod_i^n I(\tau_i)\right] \exp\left[-\int I(\tau)\,\eta(\tau)\,\phi(\tau)\,\mathrm{d}\tau\right]$$

The ‘log’ of the likelihood function now has the form,

$$\ln \mathscr{L} = \sum_i^n \ln I(\tau_i) \ - \int I(\tau)\ \eta(\tau)\ \phi(\tau)\ \mathrm{d}\tau$$

We shall adopt the following shorthand notation

$$\alpha = \{\varepsilon k; \chi\, m\} \quad \text{and} \quad \alpha' = \{\varepsilon k; \chi'\, m'\}$$

and write

$$I(\tau) = \sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\ D_\alpha(\tau)\ D^*_{\alpha'}(\tau)$$

The so-called ‘experimental’ normalization integral is given by

$$\Psi^x_{\alpha\,\alpha'} = \int \left[D_\alpha(\tau)\ D^*_{\alpha'}(\tau)\right]\ \eta(\tau)\ \phi(\tau)\ \mathrm{d}\tau$$

so that

$$\ln \mathscr{L} = \sum_i^n \ln\Big[\sum_{\alpha\alpha'} V_\alpha\ V^*_{\alpha'}\ D_\alpha(\tau_i)\ D^*_{\alpha'}(\tau_i)\Big] - \sum_{\alpha\alpha'} V_\alpha\ V^*_{\alpha'}\ \Psi^x_{\alpha\,\alpha'}$$

We explore the normalization for V's by setting $V = xW$, where x is independent of α,

$$\ln \mathscr{L} = \sum_i^n \ln \left[x^2 \sum_{\alpha\alpha'} W_\alpha \, W^*_{\alpha'} \, D_\alpha(\tau_i) \, D^*_{\alpha'}(\tau_i) \right] - x^2 \sum_{\alpha\alpha'} W_\alpha \, W^*_{\alpha'} \, \Psi^x_{\alpha\,\alpha'}$$

At the maximum, we should have

$$0 = \frac{\partial \ln \mathscr{L}}{\partial x^2} = \sum_i^n \left[\frac{1}{x^2} \right] - \sum_{\alpha\alpha'} W_\alpha \, W^*_{\alpha'} \, \Psi^x_{\alpha\,\alpha'}$$

so that

$$\sum_{\alpha\alpha'} V_\alpha \, V^*_{\alpha'} \, \Psi^x_{\alpha\,\alpha'} = n$$

We can define the theoretical normalization integral, with $\eta(\tau) = 1$,

$$\Psi_{\alpha\,\alpha'} = \int \left[D_\alpha(\tau) \, D^*_{\alpha'}(\tau) \right] \phi(\tau) \, \mathrm{d}\tau$$

The predicted number of events is

$$N = \sum_{\alpha\alpha'} V_\alpha \, V^*_{\alpha'} \, \Psi_{\alpha\,\alpha'}$$

$$\equiv \sum_{\alpha\alpha'} N_{\alpha\alpha'} \; , \qquad N_{\alpha\alpha'} = V_\alpha \, V^*_{\alpha'} \, \Psi_{\alpha\,\alpha'}$$

So the predicted number of events for a partial wave α is

$$N_{\alpha\alpha} = |V_\alpha|^2 \, \Psi_{\alpha\,\alpha} \; , \qquad \Psi_{\alpha\,\alpha} = \int \left|D_\alpha(\tau)\right|^2 \phi(\tau) \, \mathrm{d}\tau$$

The predicted number of events for the interference between the partial waves α and α' is

$$N_{\alpha\alpha'} + N_{\alpha'\alpha} = 2\Re\left\{V_\alpha \, V^*_{\alpha'} \, \Psi_{\alpha\,\alpha'}\right\} \; , \qquad \alpha \neq \alpha'$$