Selected Topics in Hadron Spectroscopy

Mathematical Techniques

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Review

- One-Particle States at Rest

(1)
$$U[R(\alpha,\beta,\gamma)]\,|jm\rangle = \sum_{m'} |jm'\rangle\, D^j_{m'm}(\alpha,\beta,\gamma)$$

- Relativistic One-Particle States

(2)
$$\text{Canonical:}\quad |\vec{p},jm\rangle = U[L(\vec{p}\,)]\,|jm\rangle = U[\overset{\circ}{R}(\phi,\theta,0)]\,U[L_z(p)]\,U^{-1}[\overset{\circ}{R}(\phi,\theta,0)]\,|jm\rangle$$

$$\text{Helicity:}\quad |\vec{p},j\lambda\rangle = U[\overset{\circ}{R}(\phi,\theta,0)]\,U[L_z(p)]\,|j\lambda\rangle = U[L(\vec{p}\,)]\,U[\overset{\circ}{R}(\phi,\theta,0)]\,|j\lambda\rangle$$

Canonical states transform like the states at rest.

Helicities are rotational invariants. Helicity frame: $\vec{z}_h \propto \vec{p}$ and $\vec{y}_h \propto \vec{z}\times\vec{z}_h$

- Parity and Time-Reversal Operations
- Two-Particle States

  ⋆ Construction (canonical and helicity); Normalization

  ⋆ Recoupling coefficients; Symmetry relations

- Applications
  - ⋆ Cross-section and Width formulas
  - ⋆ 2- and 3-body relativistic kinematics and phase space:

(3) $$\mathrm{d}\phi_2(1,2) = \frac{1}{(4\pi)^2}\,\frac{p}{w}\,\mathrm{d}\Omega\,; \qquad \mathrm{d}\phi_3(1,2,3) = \frac{4}{(4\pi)^5}\,\mathrm{d}R(\alpha,\beta,\gamma)\,\mathrm{d}E_2\,\mathrm{d}E_3$$

(4) $$\mathrm{d}\phi_n = \mathrm{d}\phi_\ell(c, m+1, \cdots, n)\left(\frac{\mathrm{d}w_c^2}{2\pi}\right)\mathrm{d}\phi_m(1, 2, \cdots, m)\ ,$$

  - ⋆ S-matrix for $a + b \to c + d$

- Applications—continued

⋆ 2-body decays:

Consider $|JM\rangle \rightarrow |\vec{p}; s_1\,\lambda_2\rangle + |-\vec{p}; s_2\,\lambda_2\rangle$, where $\vec{p} = p(\theta, \phi)$,

$$A^J_{\lambda_1\lambda_2}(M;\Omega) = \langle \vec{p}\lambda_1; -\vec{p}\lambda_2|\mathcal{M}|JM\rangle$$

(5)

$$= N_J F^J_{\lambda_1\lambda_2} D^{J\,*}_{M\lambda}(\phi,\theta,0)\ , \quad N_J = \sqrt{\frac{2J+1}{4\pi}}\ , \quad \lambda = \lambda_1 - \lambda_2\ ,$$

where

(6) $$F^J_{\lambda_1\lambda_2} = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a^J_{\ell s}\,(\ell 0\ s\lambda|J\lambda)\,(s_1\lambda_1\ s_2\ {-\lambda_2}|s\lambda)\ ,$$

Zemach amplitudes: $a^J_\ell \propto p^\ell$

Modern Methods: $a^J_\ell \propto F_\ell(p/p_R)$ (Blatt-Weisskopf barrier factors)

The symmetry relations are

(7) $$F^J_{\lambda_1\lambda_2} = \eta\eta_1\eta_2(-)^{J-s_1-s_2} F^J_{-\lambda_1-\lambda_2}\ , \qquad F^J_{\lambda_1\lambda_2} = (-)^J F^J_{\lambda_2\lambda_1}\ .$$

⋆ 3-body decays; Dalitx-plot analysis

- Decay Modes: Examples
  - ⋆ Two-pion decays; three-pion decays

    $\rho \longrightarrow \pi\pi$ and $\omega \longrightarrow \pi\pi\pi$ have an identical decay amplitude,
    if the 3π normal is chosen for the ω decay.

  - ⋆ Decays into $\rho\pi$ and $\omega\pi$

    ρ and ω decays described in the helicity frame
    Requires introduction of the Breit-Wigner functions for ρ and ω

  - ⋆ Decays modes with photons in the final states

    Spin-1 particles do not couple to two-photon final states.

- Density Matrix
  - ⋆ Density Matrix in the Reflectivity Basis:

    The indices $(i,j) = \{\chi\, m\}$ where $m \geq 0$ and $(i,j) = 1, \cdots, N_\epsilon$

(8) $$ {}^\epsilon\rho_{ij} = \sum_{k=1}^{K_\epsilon} {}^\epsilon V_{ik}\, {}^\epsilon V^*_{jk}, \quad \Longrightarrow \quad {}^\epsilon\rho = {}^\epsilon V\, {}^\epsilon V^\dagger, \quad \Longrightarrow \quad {}^\epsilon\rho = {}^\epsilon\rho^\dagger $$

  - ⋆ General Angular Distributions in the Reflectivity Basis:

    $\{\chi\, m\} = 1, \cdots, N_\epsilon$ and $\{\chi'\, m'\} = 1, \cdots, N_\epsilon$

(9) $$ I(\tau) \propto \sum_{\epsilon}^{2} \sum_{\substack{\chi m \\ \chi' m'}}^{N_\epsilon} {}^\epsilon\rho^{\chi\,\chi'}_{m m'}\; {}^\epsilon D^{\chi}_{m}(\tau)\; {}^\epsilon D^{\chi'\,*}_{m'}(\tau) $$

- Maximum-Likelihood Method

  ⋆ Extended Likelihood functions

(10) $$\alpha = \{\epsilon k; \chi\, m\} \quad \text{and} \quad \alpha' = \{\epsilon k; \chi'\, m'\}$$

(11) $$\ln \mathcal{L} = \sum_i^n \ln \left[\sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, D_\alpha(\tau_i)\, D^*_{\alpha'}(\tau_i) \right] - \sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, \Psi^x_{\alpha\,\alpha'}$$

where the experimental normalization integral is given by

(12) $$\Psi^x_{\alpha\,\alpha'} = \int \left[D_\alpha(\tau)\, D^*_{\alpha'}(\tau) \right] \eta(\tau)\, \phi(\tau)\, \mathrm{d}\tau$$

In terms of the full normalization integral,

(13) $$\Psi_{\alpha\,\alpha'} = \int \left[D_\alpha(\tau)\, D^*_{\alpha'}(\tau) \right] \phi(\tau)\, \mathrm{d}\tau$$

the predicted numbers of events are

(14) $$N = \sum_{\alpha\alpha'} V_\alpha\, V^*_{\alpha'}\, \Psi_{\alpha\,\alpha'}$$

The QCD Lagrangian:

$$\mathcal{L} = \cdots - \sum_q m_q \sum_i \bar{\psi}_q^i \, \psi_q^i \cdots$$

where $i =$ color index (1–3), $q =$ flavor index (1–6) and $m_q =$ ‘current-quark’ mass.

u	1–5	MeV
d	3–9	MeV
s	75–170	MeV
c	1.15–1.35	GeV
b	4.0–4.4	GeV
t	174.3 ± 5.1	GeV

$$\begin{cases} \text{flavor } SU(2): & (u, d) \\ \text{flavor } SU(3): & (u, d, s) \\ \text{flavor } SU(4): & (u, d, s, c) \end{cases}$$

Constitutent light-quark mass$= (m_d + m_u)/2 = 220$ MeV
Constitutent strange-quark mass$= m_s = 419$ MeV

S. Godfrey and N. Isgur, Phys. Rev. D32, 189 (1985)

C- and G-Parity: A new Definition and Applications

C- and G-Parity Operations:

We shall adopt a notation 'a' to stand for both the baryon number B and hypercharge Y. Anti-particles are denoted '$\bar{a}$', so that

$$a = (B, Y), \quad \bar{a} = (\bar{B}, \bar{Y}) = (-B, -Y) \tag{15}$$

In addition, we shall use y to denote $Y/2$;

$$y = \frac{Y}{2} = \frac{1}{2}(B + S), \quad Q = y + \nu \tag{16}$$

where S, Q, ν are the strangeness, the charge and the third component of isospin, respectively.

Let I be the isospin operator. Then, we have

$$[I_i, I_j] = i\,\epsilon_{ijk}\,I_k \tag{17}$$

We start with a state having an isospin σ and its third component ν which transforms according to the standard $|jm\rangle$ representation, so that

$$\begin{aligned} I_z|\sigma\nu\rangle &= \nu|\sigma\nu\rangle \\ I_\pm|\sigma\nu\rangle &= F_\pm(\nu)|\sigma\nu\pm 1\rangle \\ I^2|\sigma\nu\rangle &= \sigma(\sigma+1)|\sigma\nu\rangle \end{aligned} \tag{18}$$

where $I_\pm = I_x \pm iI_y$ and

$$F_\pm(\nu) = \sqrt{(\sigma\mp\nu)(\sigma\pm\nu+1)} \tag{19}$$

Note that $F_\pm(\nu) = F_\mp(-\nu)$. We shall require that anti-particle states transform in the same way as their particle states according to the standard representations given above.

The C operation changes a state $|a\nu\rangle$ to $|\bar{a}\ -\nu\rangle$. (We use a shorthand notation where the isospin σ is omitted from a more complete description of the state $|a\sigma\nu\rangle$.) If anti-particle states are to transform in the same way as particle states, it is necessary that one define an anti-particle through the G operation. The key point is that G is defined so that its operation does not perturb the ν quantum number.

To define the G operator, we need to first introduce a rotation by 180° around the y-axis:

(20)
$$U^2[R_y(\pi)] = (-)^{2\sigma},\; U^{-1}[R_y(\pi)] = (-)^{2\sigma}\, U[R_y(\pi)],$$
$$U[R_y(\pi)]|\sigma\nu\rangle = (-)^{\sigma-\nu}|\sigma\ -\nu\rangle$$

It will be shown later that $R_y(\pi)$ commutes with the C operator. We therefore define the G operator through

(21)
$$\mathbb{G} = \mathbb{C}\, U[R_y(\pi)] = U[R_y(\pi)]\,\mathbb{C}, \quad \mathbb{C} = (-)^{2\sigma}\, U[R_y(\pi)]\,\mathbb{G}$$

We are now ready to define an anti-particle state via

(22)
$$\mathbb{G}|a\nu\rangle = g|\bar{a}\nu\rangle$$
$$\mathbb{G}|\bar{a}\nu\rangle = \bar{g}|a\nu\rangle$$

and *require* that g and $\bar{g}$ be independent of ν and furthermore that an arbitrary isospin rotation $R(\alpha,\beta,\gamma)$ commutes with G:

(23)
$$\Big[U[R(\alpha,\beta,\gamma)],\mathbb{G}\Big] = 0$$

The action of $\mathbb{C}$ on particle and anti-particle states is

(24)
$$\mathbb{C}|a\nu\rangle = g(-)^{\sigma+\nu}|\bar{a}\, -\nu\rangle$$
$$\mathbb{C}|\bar{a}\nu\rangle = \bar{g}(-)^{\sigma+\nu}|a\, -\nu\rangle$$

It is customary to define $\mathbb{C}$ such that $\mathbb{C}^2 = I$, in which case

(25)
$$g\bar{g}(-)^{2\sigma} = 1$$

For hadrons, we shall define g and $\bar{g}$ via

(26)
$$g = \eta(-)^{y+\sigma}, \quad \bar{g} = \eta(-)^{\bar{y}+\sigma}$$

while for quarks,

(27)
$$g = \eta(-)^{B+y+\sigma}, \quad \bar{g} = \eta(-)^{\bar{B}+\bar{y}+\sigma}$$

Note that the exponents in these expressions are always integers.

The quantity F defined by

(28) $$\frac{1}{2}F = B + y = \frac{1}{2}(3B + S)$$

will be termed the '*intrinsic flavor*' of a particle. Note that the intrinsic flavor is always an integer, as shown in the following table:

states	u	d	s	π	η	K	p	n	Λ	Σ	Ξ	Δ	Ω^-
F	1	1	0	0	0	1	3	3	2	2	1	3	0

It is seen that the intrinsic flavor of an anti-particle is the negative of that of the particle, i.e. $\bar{F} = -F$. With these definitions, we can make η a real number and let it take on values of $+1$ or-1, so that $\eta^2 = +1$. Then, we have, since $\mathbb{C}^2 = +1$ and $g\bar{g}(-)^{2\sigma} = +1$,

(29) $$\mathbb{G}^2 = (-)^{2\sigma}$$

conforming to the standard expressions.

From the actions of the $\mathbb{C}$ on $|a\nu\rangle$ and $|\bar{a}\nu\rangle$ as defined previously, it is easy to work out the commutation relations between $\mathbb{C}$ and I;

(30) $$\{\mathbb{C}, I_x\} = \{\mathbb{C}, I_z\} = 0, \quad [\mathbb{C}, I_y] = 0$$

In other words, $\mathbb{C}$ anti-commutes with I_x and I_z while it commutes with I_y. This gives a ready justification of the definition of G-parity. We can further deduce that, since rotations commute with $\mathbb{G}$, i.e. $\mathbb{G}\, U[R(\alpha,\beta,\gamma)] = U[R(\alpha,\beta,\gamma)]\, \mathbb{G}$,

(31) $$\mathbb{C}\, U[R(\alpha,\beta,\gamma)]\, \mathbb{C}^{-1} = U[R_y(\pi)]\, U[R(\alpha,\beta,\gamma)]\, U^{-1}[R_y(\pi)]$$

This shows that the actions of I-spin rotation under charge-conjugation can be expressed in terms of I-spin $90°$ rotations.
Recapitulate:

(32) $$\begin{aligned} \mathbb{G}|a\nu\rangle &= \eta\,(-)^{y+\sigma}|\bar{a}\nu\rangle \\ \mathbb{G}|\bar{a}\nu\rangle &= \eta\,(-)^{\bar{y}+\sigma}|a\nu\rangle \\ \mathbb{C}|a\nu\rangle &= \eta\,(-)^{y-\nu}|\bar{a}\ -\nu\rangle \\ \mathbb{C}|\bar{a}\nu\rangle &= \eta\,(-)^{\bar{y}-\nu}|a\ -\nu\rangle \end{aligned}$$

For quarks, replace $y = (B+S)/2 \longrightarrow B + y = (3B+S)/2$ and $\bar{y} = (\bar{B}+\bar{S})/2 \longrightarrow \bar{B}+\bar{y} = (3\bar{B}+\bar{S})/2$. Conclude: η is the charge conjugation of the nonstrange neutral members of any meson family of $SU(3)$. [Note $G = C(-)^I$]

As an example, Consider the members of the 'pion' $SU(3)$ family, i.e. $\{\pi\}$.
We set $\eta = +1$ and find

(33)
$$\mathbb{C}\,\pi^{\pm} = -\pi^{\mp}, \qquad \mathbb{C}\,\pi^0 = +\pi^0, \qquad \mathbb{C}\,\eta = +\eta, \qquad \mathbb{C}\,\eta' = +\eta'$$
$$\mathbb{G}\,\pi = -\pi, \qquad \mathbb{G}\,\eta = +\eta$$

and

(34)
$$\mathbb{C}\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \begin{pmatrix} +K^- \\ -\bar{K}^0 \end{pmatrix}, \qquad \mathbb{C}\begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} = \begin{pmatrix} -K^0 \\ +K^+ \end{pmatrix}$$
$$\mathbb{G}\begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = \begin{pmatrix} -\bar{K}^0 \\ -K^- \end{pmatrix}, \qquad \mathbb{G}\begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix} = \begin{pmatrix} +K^+ \\ +K^0 \end{pmatrix}$$

Note that $\mathbb{C}^2 = I$ and $\mathbb{G}^2 = -I$, consistent with the usual results as applied to the states with $I = 1/2$. For the $\{\rho\}$ $SU(3)$ family, we must set $\eta = -1$, so that

(35)
$$\mathbb{C}\,\rho^{\pm} = +\rho^{\mp}, \qquad \mathbb{C}\,\rho^0 = -\rho^0, \qquad \mathbb{C}\,\omega = -\omega, \qquad \mathbb{C}\,\phi = -\phi$$
$$\mathbb{G}\,\rho = +\rho, \qquad \mathbb{G}\,\omega = -\omega$$

and

(36)
$$\mathbb{C}\begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} = \begin{pmatrix} -K^{*-} \\ +\bar{K}^{*0} \end{pmatrix}, \qquad \mathbb{C}\begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} = \begin{pmatrix} +K^{*0} \\ -K^{*+} \end{pmatrix}$$
$$\mathbb{G}\begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} = \begin{pmatrix} +\bar{K}^0 \\ +K^{*-} \end{pmatrix}, \qquad \mathbb{G}\begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} = \begin{pmatrix} -K^{*+} \\ -K^{*0} \end{pmatrix}$$

Two-Particle States:

We shall work out here the effect of C and G operations on a particle-antiparticle system in an eigenstate of total isospin, total intrinsic spin, orbital angular momentum and total spin. We use the notations I, S, ℓ and J for these quantum numbers. (Note that I was used as an isospin operator and S denoted strangeness in section2.) Each single-particle state in the two-particle center-of-mass(CM) system will be given a shorthand notation,

(37)
$$|a, +\vec{k}, \nu_1, m_1\rangle = |a, +\vec{k}, \sigma_1\nu_1, s_1m_1\rangle$$
$$|\bar{a}, -\vec{k}, \nu_2, m_2\rangle = |\bar{a}, -\vec{k}, \sigma_2\nu_2, s_2m_2\rangle$$

where $\vec{k}$ is the 3-momentum of the particle in the CM system, and σ_1 and s_1 are isospin and spin of the particles $\sigma_1 = \sigma_2 = \sigma$ and $s_1 = s_2 = s$.

The two-particle system in a given state of $|I\nu\rangle$ and $|\ell SJM\rangle$ is given by

(38)
$$|a\bar{a}\nu\rangle = \sum_{\substack{\nu_1\ \nu_2 \\ m_1\ m_2}} (\sigma_1\nu_1\sigma_2\nu_2|I\nu)(s_1m_1s_2m_2|Sm_s)(Sm_s\ell m|JM)$$
$$\times \int \mathrm{d}\vec{k}\ Y^\ell_m(\vec{k})\ |a, +\vec{k}, \nu_1, m_1\rangle|\bar{a}, -\vec{k}, \nu_2, m_2\rangle$$

where $Y^\ell_m(\vec{k})$ is the usual spherical harmonics.

We note

$$\mathbb{C}|a,+\vec{k},\nu_1,m_1\rangle|\bar{a},-\vec{k},\nu_2,m_2\rangle = (-)^{\nu_1+\nu_2}|\bar{a},+\vec{k},-\nu_1,m_1\rangle|a,-\vec{k},\ -\nu_2,m_2\rangle$$
$$= (-)^{\nu_1+\nu_2+2s}|a,-\vec{k},\ -\nu_2,m_2\rangle|\bar{a},+\vec{k},\ -\nu_1,m_1\rangle$$
$$\mathbb{G}|a,+\vec{k},\nu_1,m_1\rangle|\bar{a},-\vec{k},\nu_2,m_2\rangle = (-)^{2\sigma}|\bar{a},+\vec{k},\nu_1,m_1\rangle|a,-\vec{k},\nu_2,m_2\rangle$$
$$= (-)^{2\sigma+2s}|a,-\vec{k},\nu_2,m_2\rangle|\bar{a},+\vec{k},\nu_1,m_1\rangle$$

where the second lines have been derived by interchanging two wave functions, which brings in a factor $(-)^{2s}$, positive for mesons and negative for fermions.

The effect of $\mathbb{C}$ and $\mathbb{G}$ on the two-particle states can now be worked out. By interchanging the subscripts 1 and 2 and by the operation $\vec{k} \to -\vec{k}$, we obtain

(39)
$$\mathbb{C}|a\bar{a}\nu\rangle = (-)^{\ell+S+\nu}\ |a\bar{a}\ \ -\nu\rangle$$
$$\mathbb{G}|a\bar{a}\nu\rangle = (-)^{\ell+S+I}\ |a\bar{a}\nu\rangle$$

where we have used the relationship $Y^{\ell}_{m}(-\vec{k}) = (-)^{\ell}\, Y^{\ell}_{m}(\vec{k})$ and the following formulas for the Clebsch-Gordan coefficient

$$(\sigma_2 - \nu_2 \sigma_1 - \nu_1|I\nu) = (\sigma_1\nu_1\sigma_2\nu_2|I\nu)$$
$$(\sigma_2\nu_2\sigma_1\nu_1|I\nu) = (-)^{I-2\sigma}\,(\sigma_1\nu_1\sigma_2\nu_2|I\nu), \quad \sigma_1 = \sigma_2 = \sigma$$
$$(s_2m_2s_1m_1|Sm_s) = (-)^{S-2s}\,(s_1m_1s_2m_2|Sm_s), \quad s_1 = s_2 = s$$

We next work out the effect of the parity operation(Π) on the two-particle states. Since antifermions have opposite intrinsic parities to those of their fermion partners, the Π operation brings in the factor $(-)^{2s}$. In addition, the 3-momentum $\vec{k}$ changes sign under the Π operation. Therefore, we have

$$\Pi|a,+\vec{k},\nu_1,m_1\rangle|\bar{a},-\vec{k},\nu_2,m_2\rangle = (-)^{2s}\,|a,-\vec{k},\nu_1,m_1\rangle|\bar{a},+\vec{k},\nu_2,m_2\rangle \tag{40}$$

So, again by using the operation $\vec{k} \to -\vec{k}$, we obtain the familiar result

$$\Pi|a\bar{a}\nu\rangle = (-)^{\ell+2s}\,|a\bar{a}\nu\rangle \tag{41}$$

It follows from (31) that a particle-antiparticle with $\nu = 0$ is in an eigenstate of $\mathbb{C}$ with its eigenvalue $(-)^{\ell+S}$. This result applies to all neutral $N\bar{N}$, $q\bar{q}$, $K\bar{K}$ and $\pi\pi$ systems, with $S = 0$ for dikaon and dipion systems. For all ν, a particle-antiparticle system has the G-parity equal to $(-)^{\ell+S+I}$. Charged $N\bar{N}$, $q\bar{q}$, $K\bar{K}$ systems have $I = 1$, so that their G-parity is $(-)^{\ell+S+1}$ (again $S = 0$ for dikaons). Since the G-parity is $+1$ for dipions, one has $\ell + I =$ even for any $\pi\pi$ system. For all ν, the intrinsic parity of a particle-antiparticle system is given by $(-)^{\ell+2s}$.

$(K\bar{K}\pi)^0$ Systems:

This case represents an example of a nontrivial application of the C- and G-parity operators introduced thus far. We start with the K^* intermediate systems. A K^* decays into a πK. For K^*'s with positive strangeness, one has

$$K^{*+} = \sqrt{\frac{2}{3}}\pi^+ K^0 - \sqrt{\frac{1}{3}}\pi^0 K^+$$
$$K^{*0} = \sqrt{\frac{1}{3}}\pi^0 K^0 - \sqrt{\frac{2}{3}}\pi^- K^+ \tag{42}$$

and for negative strangeness

$$\bar{K}^{*0} = \sqrt{\frac{2}{3}}\pi^+ K^- - \sqrt{\frac{1}{3}}\pi^0 \bar{K}^0$$
$$K^{*-} = \sqrt{\frac{1}{3}}\pi^0 K^- - \sqrt{\frac{2}{3}}\pi^- \bar{K}^0 \tag{43}$$

One uses a convention in which ordering of particles signifies different momenta, so that one must keep track of it with care.

It is seen that the C and G operators act on K^*'s in the following way:

(44) $$\mathbb{C}\begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} = \begin{pmatrix} -K^{*-} \\ +\bar{K}^{*0} \end{pmatrix}, \qquad \mathbb{C}\begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} = \begin{pmatrix} +K^{*0} \\ -\bar{K}^{*+} \end{pmatrix}$$

and

(45) $$\mathbb{G}\begin{pmatrix} K^{*+} \\ K^{*0} \end{pmatrix} = \begin{pmatrix} +\bar{K}^{*0} \\ +K^{*-} \end{pmatrix}, \qquad \mathbb{G}\begin{pmatrix} \bar{K}^{*0} \\ K^{*-} \end{pmatrix} = \begin{pmatrix} -K^{*+} \\ -K^{*0} \end{pmatrix}$$

Let $A_I^g(K^*)$ stand for the decay amplitude $X^0 \to (K\bar{K}\pi)^0$ where I is the isospin of the X and g its G-parity

(46) $$A_I^g(K^*) = \frac{1}{2}\left[\left(K^{*+}K^- + g\,\bar{K}^{*0}K^0\right) - (-)^I\left(K^{*0}\bar{K}^0 + g\,K^{*-}K^+\right)\right]$$

and

(47) $$\mathbb{G}\,A_I^g(K^*) = g\,A_I^g(K^*), \qquad \mathbb{C}\,A_I^g(K^*) = g(-)^I\,A_I^g(K^*)$$

Introducing the K^* decays, one sees that

(48)
$$A_I^g(K^*) = \sqrt{\frac{1}{6}} \Big\{ \left[(\pi^+ K^0)_* K^- + g(-)^I (\pi^- \bar{K}^0)_* K^+ \right] + (-)^I \left[(\pi^- K^+)_* \bar{K}^0 + g(-)^I (\pi^+ K^-)_* K^0 \right] \Big\}$$
$$- \sqrt{\frac{1}{12}} \Big\{ \left[(\pi^0 K^+)_* K^- + g(-)^I (\pi^0 K^-)_* K^+ \right] + (-)^I \left[(\pi^0 K^0)_* \bar{K}^0 + g(-)^I (\pi^0 \bar{K}^0)_* K^0 \right] \Big\}$$

We next consider two different intermediate states involving $K\bar{K}$. Let a's refer to $a_0(980)$, $a_2(1320)$ and other $I^G = 1^-$ objects, and f's stand for either $f_0(980)$, $f_2(1270)$ or other $I^G = 0^+$ states. They are given by

(49)
$$a^0 = \frac{1}{2} \left[K^+ K^- + K^0 \bar{K}^0 + (\bar{K}^0 K^0 + K^- K^+) \right]$$
$$a^- = \sqrt{\frac{1}{2}} \left[K^0 K^- + K^- K^0 \right], \quad a^+ = \sqrt{\frac{1}{2}} \left[K^+ \bar{K}^0 + \bar{K}^0 K^+ \right]$$

where $\mathbb{G} a = -a$ and $\mathbb{C} a^0 = +a^0$ as it should be.

and

(50) $$f = \frac{1}{2}\left[K^+K^- - K^0\bar{K}^0 - (\bar{K}^0K^0 - K^-K^+)\right]$$

so that $\mathbb{G}f = +f$ and $\mathbb{C}f = +f$. Again, let $A_I^g(a)$ and $A_I^g(f)$ refer to the decay amplitude for $X^0 \to a + \pi$ and $X^0 \to f + \pi$

(51) $$\begin{aligned} A_0^+(a) &= \sqrt{\frac{1}{3}}\left(\pi^+\,a^- - \pi^0\,a^0 + \pi^-\,a^+\right) \\ &= \sqrt{\frac{1}{6}}\left[\pi^+(K^0K^-)_a + \pi^+(K^-K^0)_a + \pi^-(K^+\bar{K}^0)_a + \pi^-(\bar{K}^0K^+)_a\right] \\ &- \sqrt{\frac{1}{12}}\left[\pi^0(K^+K^-)_a + \pi^0(K^0\bar{K}^0)_a + \pi^0(\bar{K}^0K^0)_a + \pi^0(K^-K^+)_a\right] \\ A_1^+(a) &= \sqrt{\frac{1}{2}}\left(\pi^+\,a^- - \pi^-\,a^+\right) \\ &= \frac{1}{2}\left[\pi^+(K^0K^-)_a + \pi^+(K^-K^0)_a - \pi^-(K^+\bar{K}^0)_a - \pi^-(\bar{K}^0K^+)_a\right] \end{aligned}$$

and

(52) $$A_1^-(f) = \frac{1}{2}\left[\pi^0(K^+K^-)_f - \pi^0(K^0\bar{K}^0)_f - \pi^0(\bar{K}^0K^0)_f + \pi^0(K^-K^+)_f\right]$$

One sees that $CA_0^+(a) = +A_0^+(a)$, $CA_1^+(a) = -A_1^+(a)$ and $CA_1^-(f) = +A_1^-(f)$.

The complete decay amplitude for $X^0 \rightarrow (K\bar{K}\pi)^0$ may now be written

(53)
$$2A = A_0^+ + A_1^+ + A_0^- + A_1^-$$

where

(54)
$$\begin{aligned} A_0^+ &= x_0^+\, A_0^+(K^*) + y_0^+\, A_0^+(a) \\ A_1^+ &= x_1^+\, A_1^+(K^*) + y_1^+\, A_1^+(a) \\ A_0^- &= x_0^-\, A_0^-(K^*) \\ A_1^- &= x_1^-\, A_1^-(K^*) + y_1^-\, A_1^-(f) \end{aligned}$$

where the superscripts $\pm$ once again specifies $g = \pm 1$ and the subscripts 0 or 1 stand for I. The variables x_I^g and y_I^g are the unknown parameters in the problem. Note that an *isoscalar* X^0 cannot couple to $\pi^0 + f$, so that one must set $y_0^- = 0$.

Consider next the amplitude corresponding to $\pi^- K_S K^+$. The complete amplitude is

$$A = \frac{1}{2\sqrt{6}} \Big\{ x_0^+ \ [(\pi^- K^+)_* K_S + (\pi^- K_S)_* K^+]_0 - x_1^+ \ [(\pi^- K^+)_* K_S + (\pi^- K_S)_* K^+]_1$$

$$+ x_0^- \ [(\pi^- K^+)_* K_S - (\pi^- K_S)_* K^+]_0 - x_1^- \ [(\pi^- K^+)_* K_S - (\pi^- K_S)_* K^+]_1 \Big\}$$

$$+ \frac{1}{4} \left\{ \sqrt{\frac{2}{3}}\, y_0^+ \ [\pi^-(K^+ K_S)_a + \pi^-(K_S K^+)_a]_0 - y_1^+ \ [\pi^-(K^+ K_S)_a + \pi^-(K_S K^+)_a]_1 \right\}$$

Similarly one finds, for the $\pi^+ K_S K^-$ amplitude,

$$A = \frac{1}{2\sqrt{6}} \Big\{ x_0^+ \ [(\pi^+ K^-)_* K_S + (\pi^+ K_S)_* K^-]_0 + x_1^+ \ [(\pi^+ K^-)_* K_S + (\pi^+ K_S)_* K^-]_1$$

$$- x_0^- \ [(\pi^+ K^-)_* K_S - (\pi^+ K_S)_* K^-]_0 - x_1^- \ [(\pi^+ K^-)_* K_S - (\pi^+ K_S)_* K^-]_1 \Big\}$$

$$+ \frac{1}{4} \left\{ \sqrt{\frac{2}{3}}\, y_0^+ \ [\pi^+(K^- K_S)_a + \pi^+(K_S K^-)_a]_0 + y_1^+ \ [\pi^+(K^- K_S)_a + \pi^+(K_S K^-)_a]_1 \right\}$$

$$\mathbb{C}\,|\pi^- K_S K^+\rangle \Longrightarrow |\pi^+ K_S K^-\rangle$$

$\{x_0^+, x_1^-, y_0^+\} \Longrightarrow C = +1$ eigenstates

$\{x_1^+, x_0^-, y_1^+\} \Longrightarrow C = -1$ eigenstates

Flavor $SU(3)$

Irreducible Representations:

J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).
G. E. Baird and L. C. Biedenharn, J. math. Phys. 4, 1449 (1963); 5, 1723 (1964).
S. U. Chung, E. Klempt, and J. K. Körner, Eur. Phys. J. A. 15, 539 (2002)

Let $D(p,q)$ be an irreducible representation characterized by two integers p and q. For a physically realizable representation, one must have $p - q = 3\,n$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. The number of basis vectors in an irreducible representation is given by the dimensionality N of the representation

(55)
$$N = (1+p)\,(1+q)\,\left[1 + \frac{1}{2}\,(p+q)\right]$$

There are two Casimir operators F^2 and G^3 with the eigenvalues f^2 and g^3. They are given by

(56)
$$f^2 = \frac{1}{3}\,\left[p^2 + q^2 + p\,q + 3\,(p+q)\right]$$
$$g^3 = \frac{1}{18}\,(p-q)\,(2p+q+3)\,(2q+p+3)$$

So an irreducible representation can be equivalently characterized by $D(f^2, g^3)$ corresponding to the two Casimir eigenvalues.

See Table I for a few examples of practical importance.

Table I: Irreducible Representations of $SU(3)$					
N	1	8	10	$\overline{10}$	27
(p, q)	$(0,0)$	$(1,1)$	$(3,0)$	$(0,3)$	$(2,2)$
(f^2, g^3)	$(0,0)$	$(3,0)$	$(6,9)$	$(6,-9)$	$(8,0)$

An eigenstate (or a wave function) belonging to an irreducible representation is given by the eigenvalues corresponding to a set of five commuting operators

$$\{F^2,\ G^3,\ Y,\ I^2,\ I_3\} \tag{57}$$

where I is the isotopic spin and Y is the hypercharge. It is conventional to use I and Y for both operators and eigenvalues. Thus, the eigenvalue for the $SU(2)$ Casimir operator I^2 is $I(I+1)$, and that for Y is just $Y = B + S$, but the eigenvalue for I_3 is denoted m here. Introduce new notations for convenience:

$$\mu = \{f^2, g^3\}, \qquad \sigma = \{Y,\ I\}, \qquad \text{and} \qquad \nu = \{Y,\ I,\ m\} \tag{58}$$

Then, the eigenstate can be given a compact notation $\phi_\nu^{(\mu)}$.

Consider a product representation $D(p_1, q_1) \otimes D(p_2, q_2)$. It can be expanded as a direct sum of irreducible representations. The eigenstates of each irreducible representation in the expansion are given by

$$\psi \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ & & \nu \end{pmatrix} = \sum_{\nu_1, \nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} \phi_{\nu_1}^{(\mu_1)} \phi_{\nu_2}^{(\mu_2)} \tag{59}$$

following the notations used previously. The subscript γ is a label which distinguishes two irreducible representations with the same (p, q) or (f^2, g^3), e.g. $\mathbf{8_1}$ and $\mathbf{8_2}$. The transformation matrix is real and orthogonal and given by

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} = \left(\begin{matrix} \mu_1 & \mu_2 \\ \sigma_1 & \sigma_2 \end{matrix} \middle| \begin{matrix} \mu_\gamma \\ \sigma \end{matrix} \right) (I_1 m_1\, I_2 m_2 | I m) \tag{60}$$

where the first element on the right-hand side is the $SU(3)$ isoscalar factor and the second element is the usual $SU(2)$ Clebsch-Gordan coefficient.

Consider a decay process $X \to a_1 + a_2$, where X is a nonstrange ($q\bar{q} + q\bar{q}$) meson with $J^P = 1^-$ and $I^G = 1^-$ or $I^G = 1^+$. So X is an isovector ($I = 1$) meson, with both $J^{PC} = 1^{-+}$ or $J^{PC} = 1^{--}$ allowed. The decay products a_1 and a_2 belong to the ground-state ${}^1S_0$ octet, i.e. $\{\pi\} = \{\pi,\ K,\ \bar{K},\ \eta\}$. We assume here that the η is a pure $SU(3)$ octet and the η' is a pure $SU(3)$ singlet. The following expansion gives relevant irreducible representations

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8_1} \oplus \mathbf{8_2} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27} \qquad (61)$$

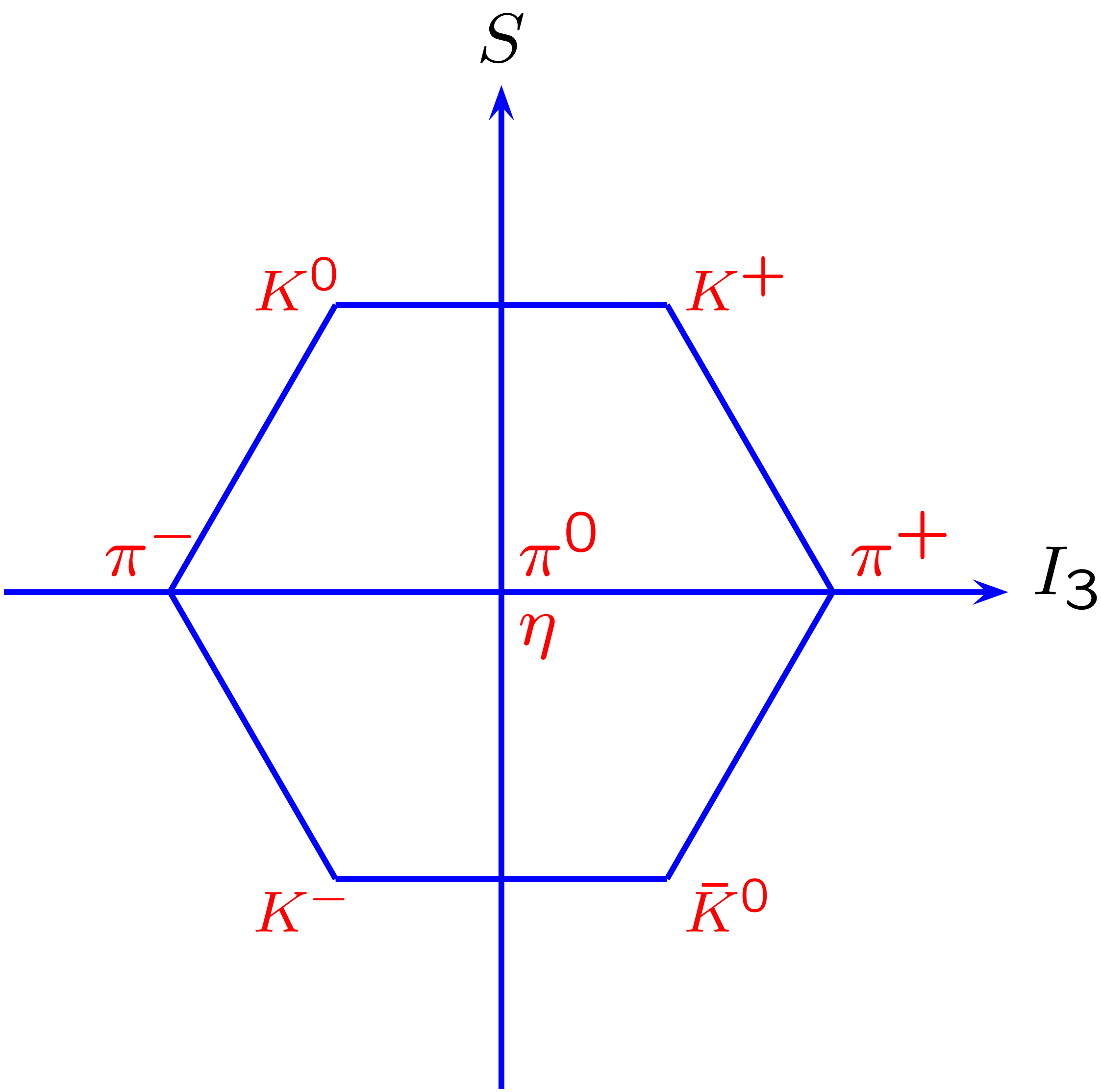
The Bose symmetrization requires that a P-wave meson couple only to antisymmetric wave functions of $SU(3)$, i.e. $\mathbf{8_2}$, $\mathbf{10}$ and $\overline{\mathbf{10}}$, as $\mathbf{1}$, $\mathbf{8_1}$ and $\mathbf{27}$ are symmetric under the interchange of a_1 and a_2.

Antisymmetric Octet (8_2): $\quad I^G = 1^+ \Longrightarrow J^{PC} = 1^{--}$

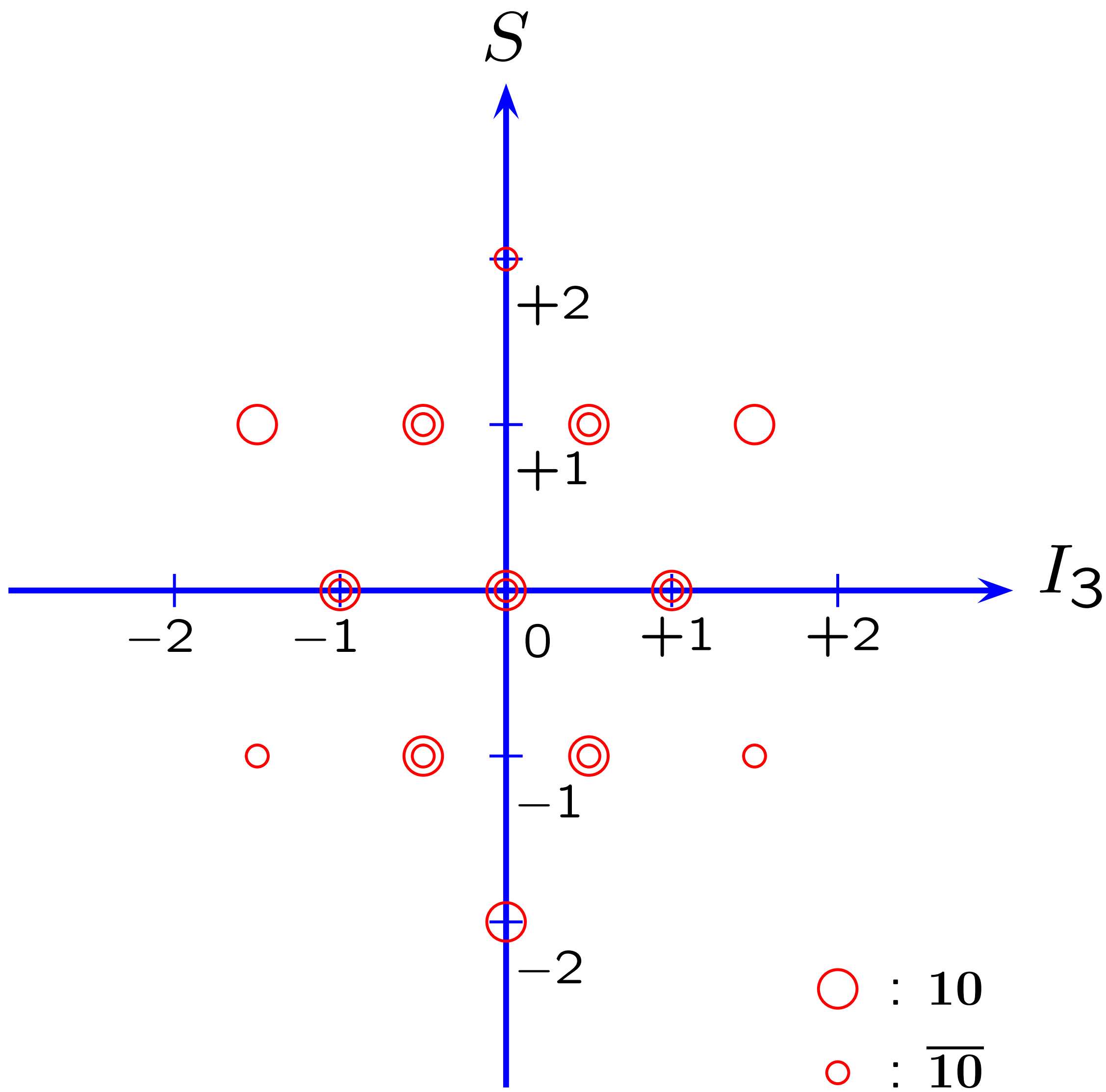
Y	I	Q	wave functions
0	1	$+1$	$\sqrt{\frac{1}{3}}\ (\pi^+\pi^0 - \pi^0\pi^+) - \sqrt{\frac{1}{6}}\ (\bar{K}^0 K^+ - K^+ \bar{K}^0)$
		0	$\sqrt{\frac{1}{3}}\ (\pi^+\pi^- - \pi^-\pi^+)$ $-\sqrt{\frac{1}{12}}\,(\bar{K}^0 K^0 - K^0 \bar{K}^0) - \sqrt{\frac{1}{12}}\,(K^- K^+ - K^+ K^-)$
		-1	$\sqrt{\frac{1}{3}}\ (\pi^0\pi^- - \pi^-\pi^0) - \sqrt{\frac{1}{6}}\ (K^- K^0 - K^0 K^-)$

An example of a self-conjugate representation: $\{\pi\}$

$\eta = +1$, $\mathbb{C}\,|\pi^0\rangle = \eta\,|\pi^0\rangle$ and $\mathbb{C}\,|a\,\nu\rangle = \eta\,(-)^{S/2-\nu}\,|\bar{a}\ -\nu\rangle$, where $I_3 = \{\nu\}$



Eigenstates in the irreducible representations of $\mathbf{10} \oplus \overline{\mathbf{10}}$:



Antisymmetric: **10**			
Y	I	Q	wave functions
0	1	$+1$	$\sqrt{\frac{1}{12}}\,(\pi^+\pi^0-\pi^0\pi^+)+\frac{1}{\sqrt{6}}\,(\bar{K}^0K^+-K^+\bar{K}^0)+\frac{1}{2}\,(\pi^+\eta-\eta\pi^+)$
		0	$\sqrt{\frac{1}{12}}\,(\pi^+\pi^--\pi^-\pi^+)$ $+\sqrt{\frac{1}{12}}\,(\bar{K}^0K^0-K^0\bar{K}^0)+\sqrt{\frac{1}{12}}\,(K^-K^+-K^+K^-)+\frac{1}{2}\,(\pi^0\eta-\eta$
		-1	$\sqrt{\frac{1}{12}}\,(\pi^0\pi^--\pi^-\pi^0)+\frac{1}{\sqrt{6}}\,(K^-K^0-K^0K^-)+\frac{1}{2}\,(\pi^-\eta-\eta\pi^-)$

Antisymmetric: $\overline{\mathbf{10}}$			
Y	I	Q	wave functions
0	1	-1	$-\sqrt{\frac{1}{12}}\,(\pi^0\pi^--\pi^-\pi^0)-\frac{1}{\sqrt{6}}\,(K^-K^0-K^0K^-)+\frac{1}{2}\,(\pi^-\eta-\eta\pi^-)$
		0	$-\sqrt{\frac{1}{12}}\,(\pi^+\pi^--\pi^-\pi^+)$ $-\sqrt{\frac{1}{12}}\,(\bar{K}^0K^0-K^0\bar{K}^0)-\sqrt{\frac{1}{12}}\,(K^-K^+-K^+K^-)+\frac{1}{2}\,(\pi^0\eta-\eta$
		$+1$	$-\sqrt{\frac{1}{12}}\,(\pi^+\pi^0-\pi^0\pi^+)-\frac{1}{\sqrt{6}}\,(\bar{K}^0K^+-K^+\bar{K}^0)+\frac{1}{2}\,(\pi^+\eta-\eta\pi^+)$

Let ϕ be the wave function for $\{\pi\pi\}$ systems. One concludes

$$\frac{1}{\sqrt{2}}\left[\phi(\mathbf{10})+\phi(\overline{\mathbf{10}})\right]_+ = \frac{1}{\sqrt{2}}\left(\pi^+\,\eta-\eta\,\pi^+\right) \quad\Longrightarrow\quad I^G(J^{PC})=1^-(1^{-+})$$

$$\frac{1}{\sqrt{2}}\left[\phi(\mathbf{10})-\phi(\overline{\mathbf{10}})\right]_+ = \frac{1}{\sqrt{6}}\left(\pi^+\,\pi^0-\pi^0\,\pi^+\right)+\frac{1}{\sqrt{3}}\left(\bar{K}^0\,K^+-K^+\,\bar{K}^0\right) \tag{62}$$

$$\Longrightarrow\quad I^G(J^{PC})=1^+(1^{--})$$

Summarize:

Consider a nonstrange isovector $X(q\bar{q}+q\bar{q})$ with the quantum numbers of a vector meson $J^P=1^-$. Its decay into $\{\pi\}+\{\pi\}$ should occur in a P wave. If $SU(3)$ is conserved in the decay,

$$\left\{\begin{array}{r} \rho(\mathbf{8_2}) : I^G(J^{PC})=1^+(1^{--}) \;\rightarrow\; \{\pi\pi\}'+\{K\bar{K}\}' \\ \rho_x(\mathbf{10}-\overline{\mathbf{10}}) : I^G(J^{PC})=1^+(1^{--}) \;\rightarrow\; \{\pi\pi\}+\{K\bar{K}\} \\ \pi_1(\mathbf{10}+\overline{\mathbf{10}}) : I^G(J^{PC})=1^-(1^{-+}) \;\rightarrow\; \pi\eta \\ \pi_1'(\mathbf{8}) : I^G(J^{PC})=1^-(1^{-+}) \;\rightarrow\; \pi\eta' \end{array}\right. \tag{63}$$

Vector mesons ($J^P = 1^-$) in $q\bar{q} + q\bar{q}$ systems:

(64) $$\{q\bar{q}\} = \mathbf{1} \oplus \mathbf{8}, \quad \{qq\} = \overline{\mathbf{3}} \oplus \mathbf{6}, \quad \{\bar{q}\bar{q}\} = \mathbf{3} \oplus \overline{\mathbf{6}}$$

and hence we must have

(65) $$\begin{aligned}\{q\bar{q}q\bar{q}\} &= (\mathbf{1} \oplus \mathbf{8}) \otimes (\mathbf{1} \oplus \mathbf{8}) = (\overline{\mathbf{3}} \oplus \mathbf{6}) \otimes (\mathbf{3} \oplus \overline{\mathbf{6}}) \\ &= 2 \times \mathbf{1} \oplus 4 \times \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}\end{aligned}$$

for a total of 81 states. There are two families of 81-plets, designated $\mathbb{V}_\zeta$ ($\zeta = \pm 1$), given by

(66) $$\begin{aligned}\{\mathbb{V}_-\} &= 61\ 1^{--} \text{ self-conjugate } \mathbf{members} \\ &+ 14\ \mathbf{strange\ members\ of\ 10\ and}\ \overline{\mathbf{10}} \\ &+\ 3\ \mathbf{1}^{--}\ \mathbf{members\ of\ 10\ and}\ \overline{\mathbf{10}} \\ &+\ 3\ \mathbf{egregious}\ \mathbf{1}^{-+}\ \mathbf{members\ of\ 10\ and}\ \overline{\mathbf{10}} \\ \{\mathbb{V}_+\} &= 61\ 1^{-+} \text{ self-conjugate } \mathbf{members} \\ &+ 14\ \mathbf{strange\ members\ of\ 10\ and}\ \overline{\mathbf{10}} \\ &+\ 3\ \mathbf{1}^{-+}\ \mathbf{members\ of\ 10\ and}\ \overline{\mathbf{10}} \\ &+\ 3\ \mathbf{egregious}\ \mathbf{1}^{--}\ \mathbf{members\ of\ 10\ and}\ \overline{\mathbf{10}}\end{aligned}$$

Let χ be the wave function for any state in $\mathbb{V}_\zeta$, and let χ^0 be its charge-neutral wave function. Then, we have

(67)
$$\mathbb{C}\,\chi^0(\mathbf{10}) = \zeta\,\chi^0(\overline{\mathbf{10}}), \quad \mathbb{C}\,\chi^0(\overline{\mathbf{10}}) = \zeta\,\chi^0(\mathbf{10})$$
$$\mathbb{G}\,\chi(\mathbf{10}) = -\zeta\,\chi(\overline{\mathbf{10}}), \quad \mathbb{G}\,\chi(\overline{\mathbf{10}}) = -\zeta\,\chi(\mathbf{10})$$

where we require $\zeta = \pm 1$. The G-parity eigenstates are

(68)
$$\chi_\pm = \frac{1}{\sqrt{2}}\left[\chi(\mathbf{10}) \mp \zeta\,\chi(\overline{\mathbf{10}})\right], \qquad \mathbb{G}\,\chi_\pm = \pm\chi_\pm$$

Define $\mathcal{H}$ to be the Hamiltonian that gives rise to the masses in the limit of exact $SU(3)$. Then , one sees that

(69)
$$[F^2, \mathcal{H}] = 0, \quad [G^3, \mathcal{H}] = 0, \quad \mathcal{H}\,|\chi_\pm\rangle = M_\pm\,|\chi_\pm\rangle$$

We see that

(70)
$$G^3\,|\chi_\pm\rangle = 9\,|\chi_\mp\rangle \quad \Longrightarrow \quad \mathcal{H}\,|\chi_\mp\rangle = M_\pm\,|\chi_\mp\rangle$$

So we conclude $M_+ = M_-$.

Recall that ϕ is the wave function defined for $\{\pi\pi\}$ systems. Then we have

$$\phi_\pm = \frac{1}{\sqrt{2}}\left[\phi(\mathbf{10}) \mp \phi(\overline{\mathbf{10}})\right], \quad \mathbb{G}|\phi_\pm\rangle = \pm|\phi_\pm\rangle \tag{71}$$

Consider now the decay $X \to \{\pi\pi\}$ in the limit of exact $SU(3)$

$$\begin{aligned} 2A_\pm &= 2\langle\phi_\pm|\mathcal{M}|\chi_\pm\rangle \\ &= \left[\langle\phi(\mathbf{10}) \mp \langle\phi(\overline{\mathbf{10}})\right]|\mathcal{M}|\left[\chi(\mathbf{10})\rangle \mp \zeta\,\chi(\overline{\mathbf{10}})\rangle\right] \\ &= \langle\phi(\mathbf{10})|\mathcal{M}|\chi(\mathbf{10})\rangle + \zeta\,\langle\phi(\overline{\mathbf{10}})|\mathcal{M}|\chi(\overline{\mathbf{10}})\rangle \\ (\mathbb{G}^\dagger\,\mathbb{G} = I,\ \zeta^2 = 1) \to &= 2\langle\phi(\mathbf{10})|\mathcal{M}|\chi(\mathbf{10})\rangle \end{aligned} \tag{72}$$

So we conclude $A_+ = A_-$ or $\langle\phi_+|\mathcal{M}|\chi_+\rangle = \langle\phi_-|\mathcal{M}|\chi_-\rangle$.

Summarizing, in the limit of exact $SU(3)$, the observation of $I^G(J^{PC}) = 1^-(1^{-+})$ $\pi_1(1400) \to \pi\eta$ implies that the $\pi_1(1400)$ must belong to the $\mathbf{10} \oplus \overline{\mathbf{10}}$ representation of $\mathbb{V}_\zeta$. And there must exist its partner $I^G(J^{PC}) = 1^-(1^{--})$ $\rho_x(1400) \to \{\pi\pi\} + \{K\bar{K}\}$ at the same mass and with the same decay strength. For example, we must have

$$g^2\left(\pi_1^+(1400) \to \pi^+\eta\right) = \frac{1}{3}g^2\left(\rho_x^+(1400) \to \pi^+\pi^0\right) + \frac{2}{3}g^2\left(\rho_x^+(1400) \to K^+\bar{K}^0\right)$$

where g^2 is the coupling constant squared.

Classification of a general decay $X \to \{\pi\} + \{\pi\}$:

$SU(3)$ Multiplet	J^{PC}	Composition
Singlet (1)	even^{++}	$q\bar{q}$, $q\bar{q} + \text{gluon}$, $q\bar{q} + q\bar{q}$
Symmetric Octet (8_1)	even^{++}	$q\bar{q}$, $q\bar{q} + \text{gluon}$, $q\bar{q} + q\bar{q}$
Antisymmetric Octet (8_2)	odd^{--}	$q\bar{q}$, $q\bar{q} + \text{gluon}$, $q\bar{q} + q\bar{q}$
Multiplet 20 ($10 + \overline{10}$)	odd^{-+}	$q\bar{q} + q\bar{q}$
Multiplet 20 ($10 - \overline{10}$)	odd^{--}	$q\bar{q} + q\bar{q}$
Multiplet 27	even^{++}	$q\bar{q} + q\bar{q}$

Mulitplet 20:

Quantum Numbers	Multiplicity	I	S
$J^P = 1^-$	8	3/2	± 1
$J^P = 1^-$	4	1/2	± 1
$J^P = 1^-$	2	0	± 2
$J^{PC} = 1^{--}$	3	1	0
$J^{PC} = 1^{-+}$	3	1	0

Gluons in QCD

Quantum chromodynamics (QCD), the field theory of strong interactions for colored quarks and gluons, is based on the group SU(3) of color. The Lagrangian is

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + i\bar{\psi}^i_k \gamma^\mu (D_\mu)_{ij} \psi^j_k - m_k \bar{\psi}^i_k \psi^i_k - \frac{1}{(8\pi)^2} \theta_{\rm QCD}\, \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} \tag{73}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu$$
$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - \frac{i}{2} g_s \lambda^a_{ij} A^a_\mu \tag{74}$$

Here the ψ^i_k is the 4-component Dirac spinor for each quark with color index $i = \{1,2,3\}$ and flavor index $k = \{1,6\}$, and A^a_μ is the gluon field with the index $a = \{1,8\}$. g_s is the QCD coupling constant, i.e. $\alpha_s = g_s^2/(4\pi)$; the f_{abc} is the usual structure constant of the SU(3) algebra, and λ^a_{ij} is the generator of the corresponding SU(3) transformations, so it is a 3×3 matrix with $a = \{1,8\}$ and $i,j = \{1,2,3\}$; and m_k, again with $k = \{1,6\}$, is the current-quark mass introduced in the previous section.

QCD is a gauge field theory with just two parameters g_s and $\theta_{\rm QCD}$.

The Flux-Tube Model

N. Isgur and J. Paton, Phys. Rev. D **31** (1985) 2910;
N. Isgur, R. Kokoski and J. Paton, Phys. Rev. Lett. **54** (1985) 869;
F. E. Close and P. R. Page, Nucl. Phys. B **443** (1995) 233;
T. Barnes, F. E. Close and E. S. Swanson, Phys. Rev. D **52** (1995) 5242.

Let L and S be the orbital angular momentum and the total intrinsic spin of a $q\bar{q}$ system. Let s_1 and s_2 be the spins of q and $\bar{q}$, i.e. $s_1 = s_2 = 1/2$. Then, one readily finds

(75) $$\vec{J} = \vec{L} + \vec{S} \qquad \text{and} \qquad \vec{S} = \vec{s}_1 + \vec{s}_2$$

and

(76) $$P = (-)^{L+1}, \quad C = (-)^{L+S}, \quad PC = (-)^{S+1}$$

and the wave function in a state of J, L and S is given by

(77) $$|JMLS\rangle = \sqrt{\frac{2L+1}{4\pi}} \sum_{\substack{m_1\, m_2 \\ m_s\, m}} (s_1 m_1\, s_2 m_2 | S m_s)\, (S m_s\, L M_L | JM) \times \int \mathrm{d}\Omega\, D^{L\,*}_{M_L 0}(\phi, \theta, 0)\, |\Omega, s_1 m_1\, s_2 m_2\rangle$$

where M is the z-component of spin J in a coordinate system given in the rest frame of $q\bar{q}$ and $\Omega = (\theta, \phi)$ specifies the direction of the breakup momentum in this rest frame.

In the flux-tube model pioneered by Isgur and Paton and an excited gluon in $q\bar{q}+g$ is described by two transverse polarization states of a string, which may be taken to be clockwise and anticlockwise about the $q\bar{q}$ axis. Let $n_m^{(+)}$ $[n_m^{(-)}]$ be the number of clockwise (anticlockwise) phonons in the mth mode of string excitation.

Two important quantities are defined from these

$$\Lambda = \sum_m \left[n_m^{(+)} - n_m^{(-)} \right], \qquad N = \sum_m m \left[n_m^{(+)} + n_m^{(-)} \right] \tag{78}$$

It is clear from the definition that Λ is the helicity of the flux tube along the $q\bar{q}$ axis. The number N is a new quantity; it will be given the name 'phonon number'—as it represents the sum of all the phonons in the problem weighted by the mode number m. The wave function in the flux-tube model is given by

$$|JMLS\,\{n_m^{(+)}, n_m^{(-)}\}\,N\,\Lambda\rangle = \sqrt{\frac{2L+1}{4\pi}} \sum_{\substack{m_1\,m_2\\ m_s\,M_L}} (s_1m_1\,s_2m_2|Sm_s)\,(Sm_s\,LM_L|JM)$$
$$\times \int \mathrm{d}\Omega\, D^{L\,*}_{M_L\Lambda}(\phi,\theta,0)\,|\Omega, s_1m_1;\, s_2m_2;\, \{n_m^{(+)}, n_m^{(-)}\}\,N\,\Lambda\rangle \tag{79}$$

There are a set of five rotational invariants specifying a state in the flux-tube model, i.e. J, L, S, N and Λ. Note $|\Lambda| \leq L$.

The relevant quantum numbers are

(80) $$P = \eta_0 \, (-)^{L+\Lambda+1}, \quad C = \eta_0 \, (-)^{L+S+\Lambda+N}, \quad PC = (-)^{S+N+1}$$

where $\eta_0 = \pm 1$. Remark: if $n_m^{(+)} = n_m^{(-)}$, then $\Lambda = 0$ and $\eta_0 = +1$. Examples are given in tabular form. The lowest-order gluonic hybrid mesons are

$n_1^{(+)} = 1$ and $n_1^{(-)} = 0$; $N = 1$ and $\Lambda = 1$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q}+g)$
1	0	$^1P_1$	1^{+-}	1^{++} 1^{--}
1	1	^3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{-+}, 1^{-+}, 2^{-+}$ $0^{+-}, 1^{+-}, 2^{+-}$
2	0	$^1D_2$	2^{-+}	2^{++} 2^{--}
2	1	^3D_J	$1^{--}, 2^{--}, 3^{--}$	$1^{-+}, 2^{-+}, 3^{-+}$ $1^{+-}, 2^{+-}, 3^{+-}$

An example of $n_m^{(+)} = n_m^{(-)}$ for $m = 1$ is given below:

$n_1^{(+)} = 1$ and $n_1^{(-)} = 1$; $N = 2$ and $\Lambda = 0$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q}+g)$
0	0	$^1S_0$	0^{-+}	0^{-+}
0	1	$^3S_1$	1^{--}	1^{--}
1	0	$^1P_1$	1^{+-}	1^{+-}
1	1	^3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{++}, 1^{++}, 2^{++}$

Another case of interest:

$n_2^{(+)} = 1$ and $n_2^{(-)} = 0$; $N = 2$ and $\Lambda = 1$

L	S	$^{2S+1}L_J(q\bar{q})$	$J^{PC}(q\bar{q})$	$J^{PC}(q\bar{q}+g)$
1	0	$^1P_1$	1^{+-}	1^{+-} 1^{-+}
1	1	^3P_J	$0^{++}, 1^{++}, 2^{++}$	$0^{++}, 1^{++}, 2^{++}$ $0^{--}, 1^{--}, 2^{--}$