

Selected Topics in Hadron Spectroscopy

Mathematical Techniques

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Spin Formalisms

Applications

We are now ready to apply results of the previous section to a few physical problems of practical importance. As a first application, we shall write down the invariant transition amplitude for two-body reactions and derive the partial-wave expansion formula. We do this in the helicity basis, following the derivation given in the “classic” paper by [Jacob and Wick](#). Our main purpose in this exercise is to show how the particular normalization of single-particle states influences the precise definition of the invariant amplitudes and the corresponding cross-section formula.

Next, we shall discuss the general two-body decays of resonances and give the symmetry relations satisfied by the decay amplitude, as well as the coupling formula which connects the helicity decay amplitude to the partial-wave amplitudes. Finally, we take up the discussion of the spin density matrices, introduce the multipole parameters, and then expand the angular distribution for two-body decays in terms of the multipole parameters.

Cross Section:

Consider a reaction

$$(1) \quad a + b \rightarrow 1 + 2 + \cdots + n .$$

In the over-all c.m. system, let w_0 be the c.m. energy and $p_i(p_f)$ the over-all four-momentum in the **initial (final)** state. In terms of the invariant amplitude \mathcal{M}_{fi} ,

$$(2) \quad d\sigma = \frac{1}{(2E_a)(2E_b) \beta_{\text{rel}}} |\mathcal{M}_{fi}|^2 d\phi_n(1, 2, \cdots, n) = \frac{1}{4\mathcal{F}} |\mathcal{M}_{fi}|^2 d\phi_n(1, 2, \cdots, n) ,$$

where \mathcal{F} is the flux factor, which in the over-all c.m. is given by

$$(3) \quad \begin{aligned} \mathcal{F} &= E_a E_b \beta_{\text{rel}} = [(p_a \cdot p_b)^2 - (w_a w_b)^2]^{1/2} \\ &= p_a w_b \text{ (stationary target)} = p_i \sqrt{s} \text{ (CM system)} = p_L \sqrt{s} \cos(\theta_L/2) \text{ (pp Collider)} \end{aligned}$$

and $d\phi_n$ is the n -body phase space:

$$(4) \quad d\phi_n(1, 2, \cdots, n) = (2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{k=1}^n \tilde{d}p_k, \quad \tilde{d}p_k = \frac{d^3\vec{p}_k}{(2\pi)^3 (2E_k)} .$$

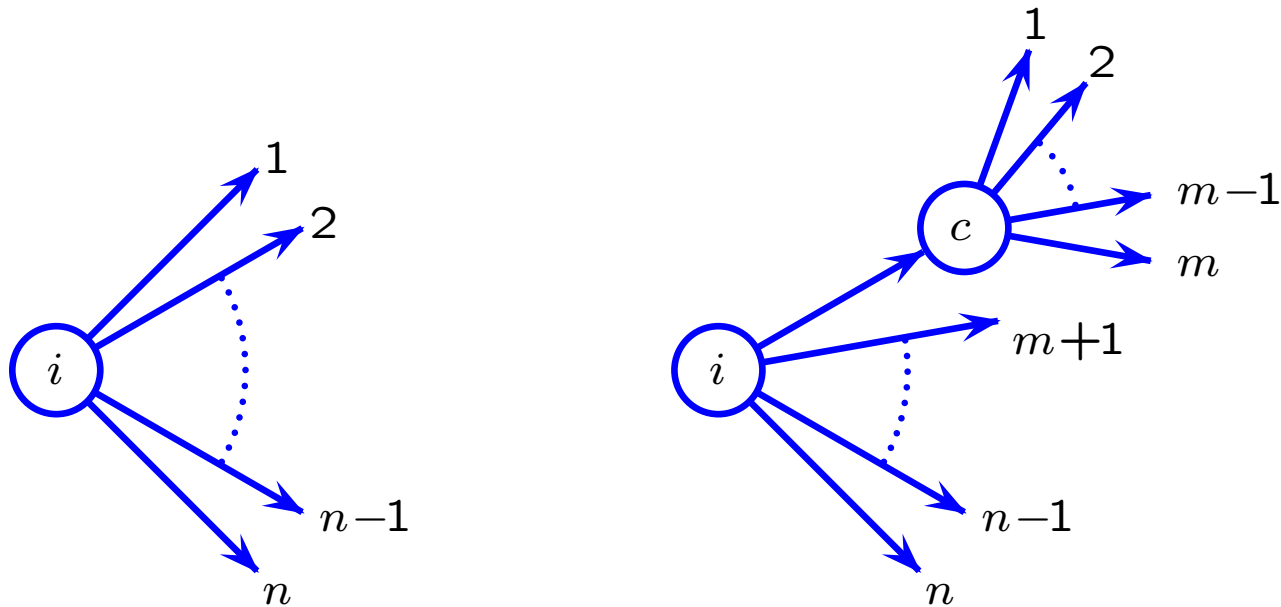
$\tilde{d}p_k$ is the invariant volume element of the k^{th} particle.

Phase Space:

The phase-space formula may be broken up into two factors as follows:

$$(5) \quad d\phi_n = d\phi_\ell(i \rightarrow c, m+1, \dots, n) \left(\frac{dw_c^2}{2\pi} \right) d\phi_m(c \rightarrow 1, 2, \dots, m),$$

where $\ell + m = n + 1$ and c denotes a system consisting of particles **1 to m** , its effective mass being w_c . The expression above may be termed the **Cluster Decomposition Formula**. Refer to the following figure:



Here the figure on the left side refer to $d\phi_n(1, 2, \dots, n)$, while that on the right side refer to the cluster decomposition formula above.

After repeated application of the formula above and using the explicit expression for the two-body phase space, we may express the n -body phase-space succinctly as follows:

$$(6) \quad d\phi_n = \frac{1}{2^n} \cdot \frac{(2\pi)^4}{(2\pi)^{3n}} \cdot \frac{p_0}{w_0} d\Omega_0 \prod_{k=0}^{n-2} \{p dw d\Omega\}_k, \quad d\phi_2(1, 2) = \frac{1}{(4\pi)^2} \frac{p}{w} d\Omega.$$

where w is the effective mass of the particles 1 and 2; and P and Ω denote the magnitude and direction of the relative momentum in the (1,2) rest frame. Note that **we must set $n \geq 2$ and $\{\dots\}_0 = 1$** . Modify the formula above to include **3-body** decays:

$$(7) \quad d\phi_n = \frac{1}{2^n} \cdot \frac{(2\pi)^4}{(2\pi)^{3n}} \cdot \frac{p_0}{w_0} d\Omega_0 \prod_{k=0}^{n_2} \{p dw d\Omega\}_k \prod_{\ell=0}^{n_3} \{w' dw' dR dE dE'\}_\ell$$

where $n_2 \geq 0$, $n_3 \geq 0$ and $n = n_2 + 2n_3 + 2 \geq 2$. Again, note $n \geq 2$ and $\{\dots\}_0 = 1$.

3-Body Phase Space:

The 3-body phase-space formula is

$$(8) \quad d\phi_3(1, 2, 3) = \frac{1}{2^3} \cdot \frac{1}{(2\pi)^5} \cdot \frac{p_0}{w_0} d\Omega_0 p'_1 dw_{12} d\Omega'$$

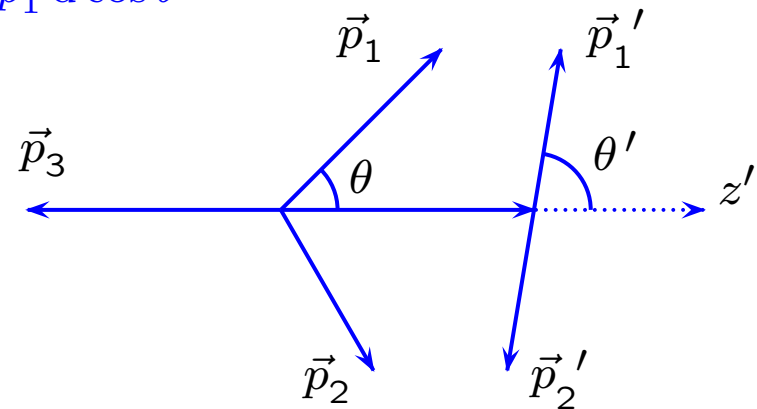
$$\rightarrow \frac{4}{(4\pi)^5} dR(\alpha, \beta, \gamma) \frac{p_3}{w_0} p'_1 d\cos\theta' \frac{dw_{12}^2}{2w_{12}}$$

where $p = p_1 + p_2 + p_3$ and $p^2 = w_0^2$; and the **primes in p'_1 and $\cos\theta'$** indicate that they are evaluated in the **(12)-RF**. But one sees that, in the (12)-RF,

$$(9) \quad w_{13}^2 = (p_1 + p_3)^2 = w_1^2 + w_3^2 + 2E'_1 E'_3 + 2p'_3 p'_1 \cos\theta'$$

so that, for a **fixed** w_{12} ,

$$(10) \quad dw_{13}^2 = 2p'_3 p'_1 d\cos\theta'$$



One needs to relate p'_3 to p_3 . For the purpose, write $p = p_{12} + p_3$ evaluate the 4-momentum square in the (12)-RF

$$(11) \quad 2w_{12} E'_3 = (w_0^2 - w_{12}^2) - w_3^2$$

and square it again

$$(12) \quad 4w_{12}^2 p_3'^2 = (w_0^2 - w_{12}^2)^2 + w_3^4 - 2(w_0^2 + w_{12}^2)w_3^2$$

Now, start with $p - p_3 = p_{12}$ and evaluate it in the overall RF

$$(13) \quad 2w_0 E_3 = (w_0^2 - w_{12}^2) + w_3^2$$

and square it

$$(14) \quad 4w_0^2 p_3^2 = (w_0^2 - w_{12}^2)^2 + w_3^4 - 2(w_0^2 + w_{12}^2)w_3^2$$

So one concludes

$$(15) \quad p'_3 = \left(\frac{w_0}{w_{12}} \right) p_3$$

So the 3-body phase-space formula becomes

$$(16) \quad d\phi_3(1, 2, 3) = \frac{4}{(4\pi)^5} dR(\alpha, \beta, \gamma) \left(\frac{dw_{13}^2 dw_{12}^2}{4w_0^2} \right) \rightarrow \text{Daltix-plot variables !}$$

This can be recast into a simple form involving energies in the overall RF. For the purpose, note

$$(17) \quad \begin{aligned} (p_{13} = p - p_2) &\rightarrow w_{13}^2 = w_0^2 + w_2^2 - 2w_0 E_2 \\ (p_{12} = p - p_3) &\rightarrow w_{12}^2 = w_0^2 + w_3^2 - 2w_0 E_3 \end{aligned}$$

where E_2 and E_3 are evaluated in the overall RF. Finally, one obtains a simple formula for the 3-body phase space

$$(18) \quad d\phi_3(1, 2, 3) = \frac{4}{(4\pi)^5} dR(\alpha, \beta, \gamma) dE_2 dE_3$$

Two-body Kinematics:

Start with $p = p_1 + p_2$ where $p^2 = w^2$, $p_1^2 = w_1^2$ and $p_2^2 = w_2^2$. Calculate energies for 1 and 2 in the (12)-RF:

$$(19) \quad \begin{aligned} (p - p_1 = p_2) &\rightarrow 2w E_1 = w^2 + w_1^2 - w_2^2 \\ (p - p_2 = p_1) &\rightarrow 2w E_2 = w^2 + w_2^2 - w_1^2 \end{aligned}$$

Note this gives $w = E_1 + E_2$. Now take the squares and solve for p^2

$$(20) \quad \begin{aligned} 4w^2 p^2 &= w^4 + w_1^4 + w_2^4 - 2 [(w w_1)^2 + (w w_2)^2 + (w_1 w_2)^2] \\ &= w^4 + (w_1 + w_2)^2 (w_1 - w_2)^2 - w^2 [(w_1 + w_2)^2 + (w_1 - w_2)^2] \\ &= [w^2 - (w_1 + w_2)^2] [w^2 - (w_1 - w_2)^2] \\ &= (w + w_1 + w_2)(w - w_1 - w_2)(w + w_1 - w_2)(w - w_1 + w_2) \end{aligned}$$

S -matrix for two-body reactions:

Let us denote a two-body reaction by

$$(21) \quad a + b \rightarrow c + d$$

with \vec{p}_a , s_a , λ_a , and η_a standing for the momentum, spin, helicity, and the intrinsic parity of the particle a , etc. Let w_0 denote the **centre-of-mass (c.m.) energy** and let \vec{p}_i (\vec{p}_f) be the **c.m. momentum** of the particle a (c). The invariant S -matrix element for the reaction may be written, in the over-all c.m. system,

$$(22) \quad \begin{aligned} \langle \vec{p}_c \lambda_c; \vec{p}_d \lambda_d | S | \vec{p}_a \lambda_a; \vec{p}_b \lambda_b \rangle &= \langle \vec{p}_f \lambda_c; -\vec{p}_f \lambda_d | S | \vec{p}_i \lambda_a; -\vec{p}_i \lambda_b \rangle \\ &= (4\pi)^2 \frac{w_0}{\sqrt{p_f p_i}} \langle \Omega \lambda_c \lambda_d | S | 00 \lambda_a \lambda_b \rangle, \end{aligned}$$

where we have used with the normalization constant given previously, and we have fixed the direction \vec{p}_i at the spherical angles $(0, 0)$ and \vec{p}_f at $\Omega = (\theta, \phi)$. Because of the invariant normalization of the one-particle states, the absolute square of the amplitude summed over the helicities λ_a , λ_b , etc., is a Lorentz invariant quantity. It is in this sense that formula above is referred to as the “invariant S matrix”. Due to the energy-momentum conservation, one may write

$$(23) \quad \langle \Omega \lambda_c \lambda_d | S | 00 \lambda_a \lambda_b \rangle = (2\pi)^4 \delta^{(4)}(p_c + p_d - p_a - p_b) \langle \Omega \lambda_c \lambda_d | S(w_0) | 00 \lambda_a \lambda_b \rangle.$$

If we define the T operator via $S = 1 + iT$, it is clear that we may write down the T -matrix in the same way, simply replacing S by T . Now, the **invariant** transition amplitude \mathcal{M}_{fi} is defined from the T matrix by

$$(24) \quad (2\pi)^4 \delta^{(4)}(p_c + p_d - p_a - p_b) \mathcal{M}_{fi} = \langle p_c \lambda_c; p_d \lambda_d | T | p_a \lambda_a; p_b \lambda_b \rangle$$

or

$$(25) \quad \mathcal{M}_{fi} = (4\pi)^2 \frac{w_0}{\sqrt{p_f p_i}} \langle \Omega \lambda_c \lambda_d | T(w_0) | 00 \lambda_a \lambda_b \rangle .$$

The cross section for $a + b \rightarrow c + d$ can be cast into

$$(26) \quad \frac{d\sigma}{d\Omega} = \frac{p_f}{p_i} \left| \frac{\mathcal{M}_{fi}}{8\pi w_0} \right|^2$$

Let us now expand the transition amplitude in terms of the partial-wave amplitudes:

$$(27) \quad \begin{aligned} \langle \Omega \lambda_c \lambda_d | T(w_0) | 00 \lambda_a \lambda_b \rangle &= \sum_{JM} \langle \Omega \lambda_c \lambda_d | JM \lambda_c \lambda_d \rangle \langle JM \lambda_c \lambda_d | T(w_0) | JM \lambda_a \lambda_b \rangle \\ &\quad \times \langle JM \lambda_a \lambda_b | 00 \lambda_a \lambda_b \rangle \\ &= \frac{1}{4\pi} \sum_J (2J + 1) \langle \lambda_c \lambda_d | T^J(w_0) | \lambda_a \lambda_b \rangle D_{\lambda\lambda'}^J(\phi_0, \theta_0, 0) , \end{aligned}$$

where $\lambda = \lambda_a - \lambda_b$ and $\lambda' = \lambda_c - \lambda_d$.

If we define the “scattering amplitude” $f(\Omega)$ via

$$(28) \quad \frac{d\sigma}{d\Omega} = \frac{p_f}{p_i} \left| \frac{\mathcal{M}_{fi}}{8\pi w_0} \right|^2 \rightarrow \frac{d\sigma}{d\Omega} = |f(\Omega)|^2$$

we obtain

$$(29) \quad f(\Omega) = \frac{(p_f/p_i)^{\frac{1}{2}}}{8\pi w_0} \mathcal{M}_{fi} .$$

This formula then relates to the “**non-relativistic**” scattering amplitude $f(\Omega)$ to the **Lorentz-invariant** transition amplitude \mathcal{M}_{fi} . One sees immediately that

$$(30) \quad f(\Omega) = \frac{1}{p_i} \sum_J \left(J + \frac{1}{2} \right) \langle \lambda_c \lambda_d | T^J(w_0) | \lambda_a \lambda_b \rangle D_{\lambda\lambda'}^{J*}(\phi, \theta, 0) .$$

The partial-wave T -matrix appearing is related to the partial-wave S -matrix by

$$(31) \quad \langle \lambda_c \lambda_d | S^J(w_0) | \lambda_a \lambda_b \rangle = \delta_{fi} \delta_{\lambda_c \lambda_a} \delta_{\lambda_d \lambda_b} + i \langle \lambda_c \lambda_d | T^J(w) | \lambda_a \lambda_b \rangle ,$$

where $\delta_{fi} = 1$ for elastic scattering and zero, otherwise.

If parity is conserved in the process, it follows that the partial-wave amplitude should satisfy the following symmetry relation:

$$(32) \quad \langle -\lambda_c - \lambda_d | S^J(w_0) | -\lambda_a - \lambda_b \rangle = \eta \langle \lambda_c \lambda_d | S^J(w_0) | \lambda_a \lambda_b \rangle ,$$

where

$$\eta = \frac{\eta_c \eta_d}{\eta_a \eta_b} (-)^{s_c + s_d - s_a - s_b} .$$

Next, we examine the consequences of time-reversal invariance. Let us denote by $|i\rangle$ and $|f\rangle$ the initial and final system in a scattering process. Then, the time-reversed process takes the initial state $|Tf\rangle$ into the final state $|Ti\rangle$, so that time-reversal invariance implies the following relation for the S -matrix:

$$(33) \quad \langle f | S | i \rangle = \langle Ti | S | Tf \rangle .$$

One finds immediately

$$(34) \quad \langle \lambda_c \lambda_d | S^J(w_0) | \lambda_a \lambda_b \rangle = \langle \lambda_a \lambda_b | S^J(w_0) | \lambda_c \lambda_d \rangle ,$$

where the right-hand side refers to the process $c + d \rightarrow a + b$.

Decay Width:

Decay rates for a state a with a 4-momentum p_a and mass w decaying into n particles of 4-momenta $p_1 \dots p_n$, i.e. for a process $p_a \rightarrow p_1 + \dots + p_n$ can be written

$$(35) \quad d\Gamma = \frac{1}{2w} |\mathcal{M}_{fi}|^2 d\phi_n(1, 2, \dots, n)$$

For a two-body decay, i.e. $p_a \rightarrow p_1 + p_2$, one has

$$(36) \quad d\phi_2(1, 2) = \frac{1}{(4\pi)^2} \frac{p}{w} d\Omega \rightarrow \frac{d\Gamma}{d\Omega} = \frac{1}{2(4\pi)^2} |\mathcal{M}_{fi}|^2 \left(\frac{p}{w^2} \right) \rightarrow w^{-2} \text{ dependence}$$

Consider a $\pi\pi$ elastic scattering in a partial wave ℓ . The invariant amplitude takes on the form

$$(37) \quad \mathcal{M}_{fi} \propto \frac{w}{p} (2\ell + 1) P_\ell(\cos \theta) e^{i\delta_\ell} \sin \delta_\ell$$

$$(38) \quad \cot \delta_\ell = \frac{w_0^2 - w^2}{w_0 \Gamma(w)}$$

so that

$$(39) \quad \mathcal{M}_{fi} \propto \frac{w}{p} (2\ell + 1) P_\ell(\cos \theta) \frac{w_0 \Gamma(w)}{w_0^2 - w^2 - iw_0 \Gamma(w)} \quad \text{from unitarity}$$

Let \vec{p}_i and \vec{p}_f be the initial and final CM momenta, i.e $|\vec{p}_i| = |\vec{p}_f| = p$ and $(\vec{p}_i \cdot \vec{p}_f) = p^2 \cos \theta$. And so, if one sets

$$(40) \quad \Gamma \propto \frac{1}{w} p^{2\ell+1}$$

then one finds the **simplest** invariant amplitude

$$(41) \quad \mathcal{M}_{fi} \propto (\vec{p}_i \cdot \vec{p}_f)^\ell$$

The width formula can be written

$$(42) \quad \Gamma(w)|_{w=w_0} = \Gamma_0, \quad \Gamma(w) = \Gamma_0 \left(\frac{w_0}{w}\right) \left(\frac{p}{p_0}\right) \left[\frac{F_\ell(p)}{F_\ell(p_0)}\right]^2 \rightarrow w^{-1} \text{ dependence !}$$

where $F_\ell(p)$ is the Blatt-Weisskopf barrier factor, given by

$$(43) \quad F_0(p) = 1, \quad F_1(p) = \sqrt{\frac{z}{z+1}}, \quad F_2(p) = \sqrt{\frac{z^2}{(z-3)^2 + 9z}}$$

where $z = (p/p_r)^2$ and p_r is an additional '**scale**' parameter in the problem which is presumably close to **0.1973 GeV/c** corresponding to the length of **1 fermi**. Note that one has adopted a normalization such that $B_\ell(p) = 1$ for $z \rightarrow \infty$.

BW barrier factor: F. von Hippel and C. Quigg, Phys. Rev. 5, 624 (1972)

Two-body decays:

Let us consider a resonance of spin-parity J^η and mass w (to be called the resonance J), decaying into a two-particle system with particles 1 and 2:

$$(44) \quad J \rightarrow 1 + 2 ,$$

and let $s_1(s_2)$ and $\eta_1(\eta_2)$ denote the spin and intrinsic parity of the particle 1(2). In the rest frame of the resonance J (**JRF**), let \vec{p} be the momentum of the particle 1 with the spherical angles given by $\Omega = (\theta, \phi)$. Then, the amplitude A describing the decay of spin J with the z -component M into two particles with helicities λ_1 and λ_2 may be written

$$(45) \quad \begin{aligned} A_{\lambda_1 \lambda_2}^J(M; \Omega) &= \langle \vec{p} \lambda_1; -\vec{p} \lambda_2 | \mathcal{M} | JM \rangle \\ &= 4\pi \left(\frac{w}{p} \right)^{\frac{1}{2}} \langle \phi \theta \lambda_1 \lambda_2 | JM \lambda_1 \lambda_2 \rangle \langle JM \lambda_1 \lambda_2 | \mathcal{M} | JM \rangle \end{aligned}$$

$$A_{\lambda_1 \lambda_2}^J(M; \Omega) = N_J F_{\lambda_1 \lambda_2}^J D_{M\lambda}^{J*}(\phi, \theta, 0), \quad \lambda = \lambda_1 - \lambda_2 ,$$

The “helicity decay amplitude” F is given by

$$(46) \quad F_{\lambda_1 \lambda_2}^J = 4\pi \left(\frac{w}{p} \right)^{\frac{1}{2}} \langle JM \lambda_1 \lambda_2 | \mathcal{M} | JM \rangle .$$

Since \mathcal{M} is a rotational invariant, the helicity amplitude F can depend only on the **rotationally invariant quantities**, namely, J , λ_1 , and λ_2 .

It is easy to expand the helicity decay amplitude F in terms of the partial-wave amplitudes. Using the recoupling coefficient, we may write

$$\begin{aligned} \langle JM\lambda_1\lambda_2|\mathcal{M}|JM\rangle &= \sum_{\ell s} \langle JM\lambda_1\lambda_2|JM\ell s\rangle \langle JM\ell s|\mathcal{M}|JM\rangle \\ &= \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} (\ell 0 s\lambda|J\lambda)(s_1\lambda_1 s_2 -\lambda_2|s\lambda) \langle JM\ell s|\mathcal{M}|JM\rangle \end{aligned}$$

so that F may be expressed

$$(47) \quad F_{\lambda_1\lambda_2}^J = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1}\right)^{\frac{1}{2}} a_{\ell s}^J (\ell 0 s\lambda|J\lambda) (s_1\lambda_1 s_2 -\lambda_2|s\lambda),$$

where the partial-wave amplitude $a_{\ell s}^J$ is defined by

$$(48) \quad a_{\ell s}^J = 4\pi \left(\frac{w}{p}\right)^{\frac{1}{2}} \langle JM\ell s|\mathcal{M}|JM\rangle.$$

The normalizations have a simple relationship

$$(49) \quad \sum_{\lambda_1\lambda_2} |F_{\lambda_1\lambda_2}^J|^2 = \sum_{\ell s} |a_{\ell s}^J|^2$$

If parity is conserved in the decay, we have

$$(50) \quad F_{\lambda_1 \lambda_2}^J = \eta_1 \eta_2 (-)^{J-s_1-s_2} F_{-\lambda_1 -\lambda_2}^J ,$$

where η_1 and η_2 are the intrinsic parities of the particles 1 and 2. If the particles 1 and 2 are identical, we have to replace the state $|JM\lambda_1\lambda_2\rangle$ by its symmetrized state, so that we obtain the following symmetry relation:

$$(51) \quad F_{\lambda_1 \lambda_2}^J = (-)^J F_{\lambda_2 \lambda_1}^J .$$

Note that **J= integer** always!

It is possible to obtain a further symmetry relation on F by considering the time-reversal operations. For the purpose, let us consider the elastic scattering of particles 1 and 2 in the angular momentum state $|JM\lambda_1\lambda_2\rangle$, i.e.

$$(52) \quad \langle JM\lambda'_1\lambda'_2|T(w)|JM\lambda_1\lambda_2\rangle \equiv \langle \lambda'_1\lambda'_2|T^J(w)|\lambda_1\lambda_2\rangle ,$$

where w is the c.m. energy and coincides with the effective mass of the resonance J . Now, we make the assumption that the J^{th} partial wave for the elastic scattering of particles 1 and 2 is completely dominated by the resonance at the c.m. energy w

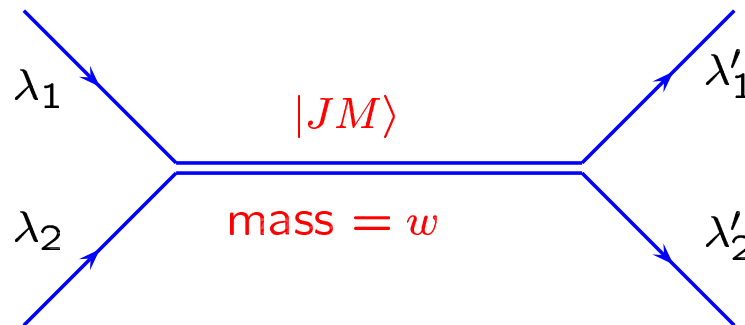


Fig. 2: Elastic scattering of particles 1 and 2, mediated by a resonance J in the s -channel.

Then, we may write

$$T(w) \sim \sum_M \mathcal{M} |JM\rangle D(w) \langle JM | \mathcal{M}^\dagger ,$$

where $D(w)$ is the Breit-Wigner function for the resonance and \mathcal{M} is an appropriate “decay operator.” We obtain

$$\langle \lambda'_1 \lambda'_2 | T^J(w) | \lambda_1 \lambda_2 \rangle \sim D(w) F_{\lambda'_1 \lambda'_2}^J F_{\lambda_1 \lambda_2}^{J*} ,$$

so that time-reversal invariance for elastic scattering implies

$$(53) \quad \langle \lambda'_1 \lambda'_2 | T^J(w) | \lambda_1 \lambda_2 \rangle = \langle \lambda_1 \lambda_2 | T^J(w) | \lambda'_1 \lambda'_2 \rangle \rightarrow F_{\lambda'_1 \lambda'_2}^J F_{\lambda_1 \lambda_2}^{J*} = F_{\lambda_1 \lambda_2}^J F_{\lambda'_1 \lambda'_2}^{J*} .$$

This means that the phase of the complex amplitude F does not depend on the helicities λ_1 and λ_2 . Therefore, we can consider F a real quantity without loss of generality:

$$(54) \quad F_{\lambda_1 \lambda_2}^J = \text{real} .$$

We emphasize that this result follows only from the assumption that **the J^{th} partial wave is dominated by the resonance J at the energy w** . This condition is fulfilled, for example, in the P -wave amplitudes of the $\pi^+\pi^-$ or $p\pi^+$ elastic scattering at the c.m. energies corresponding to ρ^0 and $\Delta(1232)$ masses, where it is known that these resonances saturate **the unitarity limit**. It is clear, however, that this condition may not be satisfied for all resonances. In this sense, the reality constraint may be considered only an “approximate” symmetry. We will show later in the discussion of the sequential decay modes that this symmetry can actually be tested experimentally.

Three-Particle Systems:

A system consisting of three particles may be treated most elegantly in the helicity basis, as was done by

M. Berman and M. Jacob, *Phys. Rev.* 139 B, 1023 (1965).

In this section, we shall first construct a three-particle system in a definite angular momentum state and then apply the formalism to a case of a resonance decaying into three particles. We will give the decay angular distribution in terms of the spin density matrix and discuss the implications of parity conservation. Finally, we will show that in a **Dalitz-plot analysis** different spin-parity states of the three-particle system do not interfere with one another.

Consider a system of three particles 1, 2, and 3. Let us use the notations s_i , η_i , λ_i , and w_i for the spin, intrinsic parity, helicity, and mass of the particle i . In the rest frame (RF) of the three particles, the momentum and energy of the particle i will be denoted by \vec{p}_i and E_i . In the RF, we define the “standard orientation” of the three-particle system, as shown in Fig. 3. this coordinate system is then the “body-fixed” coordinate system, which may be rotated by the Euler angles α , β , and γ to obtain a system with arbitrary orientation.

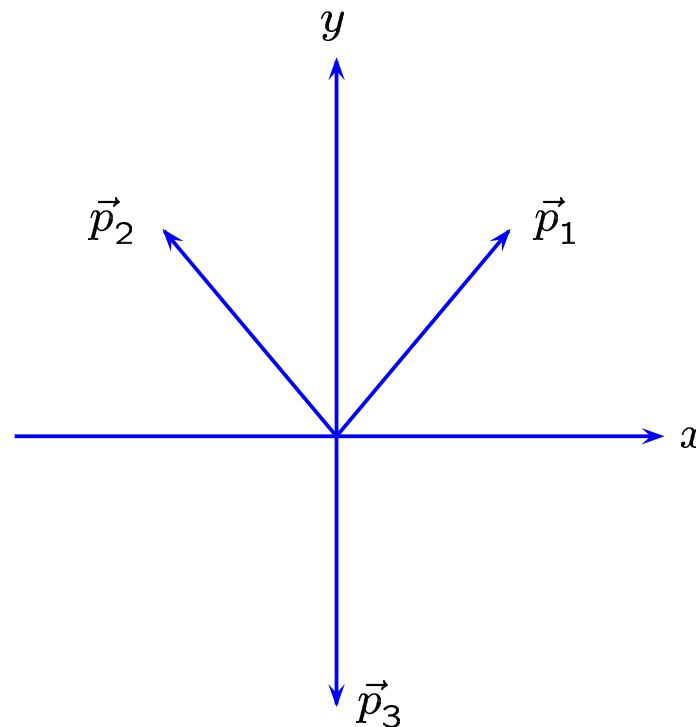


Fig. 3: Standard orientation of the three-particle rest system. Note that the y -axis is defined along the **negative direction** of \vec{p}_3 , and the z -axis along $\vec{p}_1 \times \vec{p}_2$.

A system with the standard orientation can be written

$$(55) \quad |000, E_i \lambda_i\rangle = b \prod_{i=1}^3 |\vec{p}_i s_i \lambda_i\rangle ,$$

where b is a normalization constant and the helicity basis vectors for each individual particle are given in the usual way:

$$(56) \quad |\vec{p} s_i \lambda_i\rangle = U[R_i L_z(p_i)] |s_i \lambda_i\rangle, \quad R_i = R(\phi_i, \pi/2, 0) .$$

A three-particle system with an arbitrary orientation in the RF can now be obtained by applying a rotation $R(\alpha, \beta, \gamma)$ to the state:

$$(57) \quad |\alpha\beta\gamma, E_i \lambda_i\rangle = U[R(\alpha, \beta, \gamma)] |000, E_i \lambda_i\rangle .$$

If we impose the normalization of the above states via

$$(58) \quad \langle \alpha' \beta' \gamma', E'_i \lambda'_i | \alpha \beta \gamma, E_i \lambda_i \rangle = \delta^{(3)}(R' - R) \delta(E'_1 - E_1) \delta(E'_2 - E_2) \prod_i \delta_{\lambda_i \lambda'_i}$$

we obtain easily (see Appendix C) that the normalization constant b should be chosen as follows:

$$(59) \quad b^{-1} = 8\pi^2 \sqrt{4\pi} .$$

Let us now define a state of definite angular momentum:

$$(60) \quad |JM\mu, E_i\lambda_i\rangle = \frac{N_J}{\sqrt{2\pi}} \int dR D_{M\mu}^{J*}(\alpha, \beta, \gamma) |\alpha\beta\gamma, E_i\lambda_i\rangle ,$$

where N_J is the normalization constant given before. That this state represents a state of definite angular momentum is easy to show following steps identical to those for two-body decays. Therefore, the states above transform under a rotation R' according to

$$(61) \quad U[R'] |JM\mu, E_i\lambda_i\rangle = \sum_{M'} D_{M'\mu}^J(R') |JM'\mu, E_i\lambda_i\rangle .$$

This relation also shows that, in addition to the obvious invariants E_i and λ_i , the quantity μ is also a rotational invariant.

Let us examine the transformation property of the states under parity operations

$$\begin{aligned}
 \Pi|000, E_i \lambda_i\rangle &= b \prod_i \Pi|R_i, p_i, s_i \lambda_i\rangle \\
 &= b \prod_i \eta_i e^{-i\pi s_i} |\bar{R}_i, p_i, s_i - \lambda_i\rangle \\
 (62) \qquad &= \left\{ \prod_i \eta_i e^{-i\pi s_i} \right\} U[R(\pi, 0, 0)] |000, E_i - \lambda_i\rangle,
 \end{aligned}$$

where $\bar{R}_i = R(\pi + \phi_i, \pi/2, 0) = R(\pi, 0, 0)R_i$, so that

$$(63) \qquad \Pi|\alpha\beta\gamma, E_i \lambda_i\rangle = \left\{ \prod_i \eta_i e^{-i\pi s_i} \right\} U[R(\alpha, \beta, \gamma + \pi)] |000, E_i - \lambda_i\rangle.$$

Changing the integration over γ into one over $\gamma' = \gamma + \pi$, we obtain finally

$$(64) \qquad \Pi|JM\mu, E_i \lambda_i\rangle = \eta_1 \eta_2 \eta_3 (-)^{s_1 + s_2 + s_3 + \mu} |JM\mu, E_i - \lambda_i\rangle.$$

We note that this formula is **not** the same as that given in **Berman and Jacob**. The reason for this is that their definition of one-particle helicity states involves a rotation $R(\phi, \theta, -\phi)$, instead of our convention $R(\phi, \theta, 0)$.

In order to treat the case when two of the three particles are identical, we shall work out a transformation formula for exchanging the particles 1 and 2. **The exchange operator \mathbb{P}_{12} is equivalent to performing a rotation by π around the body-fixed y -axis (see Fig. 3).** Using an identity

$$(65) \quad R(\pi + \alpha, \pi - \beta, \pi - \gamma) = R(\alpha, \beta, \gamma)R(0, \pi, 0)$$

we obtain

$$(66) \quad \mathbb{P}_{12}|\alpha\beta\gamma, E_1\lambda_1, E_2\lambda_2, E_3\lambda_3\rangle = |\pi + \alpha, \pi - \beta, \pi - \gamma, E_2\lambda_2, E_1\lambda_1, E_3\lambda_3\rangle ,$$

Combining this formula with the **defining** formula for $|JM\mu, E_i\lambda_i\rangle$, and using an identity

$$(67) \quad D_{m'm}^j(\pi + \alpha, \pi - \beta, \pi - \gamma) = (-)^{j-m} D_{m'-m}^j(\alpha, \beta, \gamma)$$

we find

$$(68) \quad \mathbb{P}_{12}|JM\mu, E_1\lambda_1, E_2\lambda_2, E_3\lambda_3\rangle = (-)^{J+\mu} |JM -\mu, E_2\lambda_2, E_1\lambda_1, E_3\lambda_3\rangle .$$

Again, this formula is not the same as that given in **Berman and Jacob**. This arises because their standard orientation for the three-particle system has been defined differently from our convention; their coordinate system has been set up with the negative x -axis along the momentum \vec{p}_3 .

Our angular momentum states are normalized according to

$$(69) \langle J' M' m u' E'_i \lambda'_i | J M \mu E_i \lambda_i \rangle = \delta_{J J'} \delta_{M M'} \delta_{\mu \mu'} \delta(E_1 - E_1) \delta(E_2 - E_2) \prod_i \delta_{\lambda_i \lambda'_i} .$$

The completeness relation is given by

$$(70) \sum_{\substack{J M \\ \mu \lambda_i}} \int |J M \mu E_i \lambda_i\rangle dE_1 dE_2 \langle J M \mu E_i \lambda_i| = I .$$

The recoupling matrix element

$$(71) \langle \alpha \beta \gamma, E'_i \lambda'_i | J M \mu, E_i \lambda_i \rangle = \frac{N_J}{\sqrt{2\pi}} D_{M\mu}^{J*}(\alpha, \beta, \gamma) \delta(E'_1 - E_1) \delta(E'_2 - E_2) \prod_1 \delta_{\lambda_i \lambda'_i} .$$

We are now ready to discuss the process in which a resonance J with spin-parity η and mass w decays into three particles 1, 2, and 3. In the rest frame of the resonance (**JRF**), let the angles (α, β, γ) describe the orientation of the three-particle system. Then, the decay amplitude may be written, with $R(\alpha, \beta, \gamma)$,

$$(72) \quad A_{\mu \lambda_i}^J(M; R) = \langle R, E_i \lambda_i | \mathcal{M} | JM \rangle = \langle R, E_i \lambda_i | JM \mu E_i \lambda_i \rangle \langle JM \mu E_i \lambda_i | \mathcal{M} | JM \rangle$$

$$A_{\mu \lambda_i}^J(M; R) = \frac{N_J}{\sqrt{2\pi}} F_{\mu}^J(E_i \lambda_i) D_{M\mu}^{J*}(R)$$

If the “decay operator” \mathcal{M} is rotationally invariant, the decay amplitude F should depend **only** on the rotational invariants, i.e.

$$(73) \quad F_{\mu}^J(E_i \lambda_i) = \langle JM \mu E_i \lambda_i | \mathcal{M} | JM \rangle .$$

If parity is conserved in the decay, we have the symmetry:

$$(74) \quad F_{\mu}^J(E_i \lambda_i) = \eta \eta_1 \eta_2 \eta_3 (-)^{s_1 + s_2 + s_3 + \mu} F_{\mu}^j(E_i - \lambda_i) .$$

And, if particles 1 and 2 are identical,

$$(75) \quad F_{\mu}^J(E_1 \lambda_1, E_2 \lambda_2, E_3 \lambda_3) = \pm (-)^{J + \mu} F_{-\mu}^J(E_2 \lambda_2, E_1 \lambda_1, E_3 \lambda_3) ,$$

where the **plus sign** holds for two identical **bosons** and the **minus sign** for **fermions**.

Dalitz-Plot Analysis:

The overall amplitude for the production and decay of a three-particle system is, for $R(\alpha, \beta, \gamma)$,

$$\sum_{JM\eta} P(JM\eta) F_{\mu}^{J\eta}(E_i\lambda_i) D_{M\mu}^{J*}(R)$$

where $P(JM\eta)$ is the **production amplitude** of state $|JM\eta\rangle$ in which η is its **intrinsic parity**. The distribution function takes on the form

$$(76) \quad I(R, E_i) \propto \sum_{\lambda_i} \sum_{\substack{JM\eta\mu \\ J'M'\eta'\mu'}} P(JM\eta) P^*(J'M'\eta') F_{\mu}^{J\eta}(E_i\lambda_i) F_{\mu'}^{J'\eta'}{}^*(E_i\lambda_i) D_{M\mu}^{J*}(R) D_{M'\mu'}^{J'}(R)$$

Integrating over $dR(\alpha, \beta, \gamma)$, we obtain

$$\int I(R, E_i) dR \propto \sum_{\lambda_i} \sum_{JM\mu} \sum_{\eta\eta'} P(JM\eta) P^*(JM\eta') F_{\mu}^{J\eta}(E_i\lambda_i) F_{\mu}^{J\eta'}{}^*(E_i\lambda_i)$$

$$(\text{parity in decay}) \rightarrow \propto \eta\eta' \sum_{\lambda_i} \sum_{JM\mu} \sum_{\eta\eta'} P(JM\eta) P^*(JM\eta') F_{\mu}^{J\eta}(E_i\lambda_i) F_{\mu}^{J\eta'}{}^*(E_i\lambda_i)$$

Conclude: **No interference** between **different J** or **η** in a Dalitz-plot analysis.

Decay Modes: Examples

Recapitulate:

Within the helicity formalism, we have seen

$$(77) \quad A_{\lambda_1 \lambda_2}^J(M; \Omega) = N_J F_{\lambda_1 \lambda_2}^J D_{M\lambda}^{J*}(\phi, \theta, 0), \quad N_J = \sqrt{\frac{2J+1}{4\pi}}, \quad \lambda = \lambda_1 - \lambda_2,$$

where

$$F_{\lambda_1 \lambda_2}^J = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1} \right)^{\frac{1}{2}} a_{\ell s}^J(\ell 0 s \lambda | J \lambda) (s_1 \lambda_1 s_2 -\lambda_2 | s \lambda),$$

Symmetry properties

$$F_{\lambda_1 \lambda_2}^J = \eta \eta_1 \eta_2 (-)^{J-s_1-s_2} F_{-\lambda_1 -\lambda_2}^J \text{ (parity)}, \quad F_{\lambda_1 \lambda_2}^J = (-)^J F_{\lambda_2 \lambda_1}^J \text{ (identical)}$$

Obtain their counterparts for the ℓs amplitude. Starting from

$$(78) \quad a_{\ell s}^J = 4\pi \left(\frac{w}{p} \right)^{\frac{1}{2}} \langle JM \ell s | \mathcal{M} | JM \rangle$$

we see that

$$\eta = \eta_1 \eta_2 (-)^\ell \text{ (parity)}, \quad \ell + s - 2s_1 = \text{even (identical)}$$

Two-pion Decays:

Two famous examples are $\rho \rightarrow \pi\pi$ and $f_2(1270) \rightarrow \pi\pi$. Let ℓ be the **spin**. Again, we use η to denote the **intrinsic parity** of the resonance decaying into two pions. The decay amplitude is

$$(79) \quad A^\ell(m; \Omega) \propto F^\ell D_{m0}^{\ell*}(\phi, \theta, 0) \propto F^\ell Y_m^\ell(\Omega)$$

From parity conservation in the decay, we must have $F^\ell = \eta (-)^\ell F^\ell$, so that

$$(80) \quad \eta (-)^\ell = +1$$

If the π 's are **identical** (i.e. the same charge), then we must $F^\ell = (-)^\ell F^\ell$. So we see that

$$(81) \quad \ell = \text{even}$$

Examples: $\rho \not\rightarrow \pi^0\pi^0$ and $f_2(1270) \rightarrow \pi^0\pi^0$.

Three-pion Decays:

We consider the decays $\omega \rightarrow \pi^+ \pi^0 \pi^-$ and $a_2(1320) \rightarrow 3\pi$. The decay amplitude is

$$(82) \quad A_\mu^J(M; R) \propto F_\mu^J(E_i) D_{M\mu}^{J*}(R)$$

From parity conservation, we see that $F_\mu^J(E_i) = \eta(-)^{\mu+1} F_\mu^J(E_i)$, so that

$$(83) \quad \eta(-)^\mu = -1$$

So we conclude that $\mu = \text{even}$ if $\eta = -1$ and $\mu = \text{odd}$ if $\eta = +1$. This shows that there is **one** amplitude ($\mu = 0$) for the ω decay

$$(84) \quad \begin{aligned} \omega \rightarrow 3\pi; \quad A_0^J(M; R) &\propto F_0^J(E_i) D_{M0}^{J*}(R), \quad J = 1 \\ \text{Integrate over } \gamma; \quad A_0^J(M; \Omega) &\propto F_0^J(E_i) D_{M0}^{J*}(\phi, \theta, 0) \propto F_0^J(E_i) Y_M^J(\Omega) \end{aligned}$$

So $\omega \rightarrow 3\pi$ decay is 'formally' **equivalent** to $\rho \rightarrow 2\pi$, if the **analyzer** is the **decay normal**. There are **two** decay amplitudes ($\mu = \pm$) for $a_2(1320) \rightarrow 3\pi$.

$$(85) \quad a_2(1320) \rightarrow 3\pi; \quad A_\pm^J(M; R) \propto F_\pm^J(E_i) D_{M\pm}^{J*}(R), \quad J = 2$$

Consider now the decays of the charged a_2 , i.e. $a_2^\pm(1320) \rightarrow \pi^\pm \pi^+ \pi^-$. Then we must have $F_\pm^J(E_1, E_2) = -F_\mp^J(E_2, E_1)$, i.e. the π 's **1 and 2** are **identical**.

Decays into $\rho\pi$ and $\omega\pi$:

Let λ be the helicity of the ρ or ω . The decay amplitude takes on the form

$$A_\lambda^J(M; \Omega) \propto F_\lambda^J D_{M\lambda}^{J*}(\phi, \theta, 0)$$

$$(86) \quad F_\lambda^J = \sum_\ell \left(\frac{2\ell + 1}{2J + 1} \right)^{\frac{1}{2}} a_\ell^J(\ell 0 s \lambda | J \lambda), \quad s = 1$$

(a) Consider the decay $a_2(1320) \rightarrow \rho\pi$.

Because of parity conservation, we must have $\ell = 2$ (D wave) only. For this case, we obtain

$$(87) \quad F_\lambda^J = \left(\frac{2\ell + 1}{2J + 1} \right)^{\frac{1}{2}} a_\ell^J(\ell 0 s \lambda | J \lambda); \quad J = \ell = 2, \quad s = 1$$

There is one **single** complex amplitude a_ℓ^J ($J = \ell = 2$) in the problem:

$$(88) \quad F_+^J = -\sqrt{\frac{1}{2}} a_\ell^J, \quad F_0^J = 0, \quad F_-^J = \sqrt{\frac{1}{2}} a_\ell^J, \quad J = \ell = 2$$

Zemach amplitudes $\rightarrow a_\ell^J \propto p^\ell \implies a_\ell^J \propto F_\ell(p/p_R)$ (**Blatt-Weisskopf** barrier factors).

Here p is the ρ momentum in the a_2 RF.

Include the decay of the $\rho \rightarrow \pi_1 + \pi_2$. Then the **overall** amplitude is

$$(89) \quad A^J(M; \Omega; w, \Omega_h) \propto \sum_{\lambda} F_{\lambda}^J D_{M\lambda}^{J*}(\phi, \theta, 0) \Delta(w) D_{\lambda 0}^{J*}(\phi_h, \theta_h, 0); \quad J = 2, \quad |\lambda| \leq 1$$

where $\Omega = (\theta, \phi)$ describes the direction of the ρ momentum in the **JRF**, and $\Omega_h = (\theta_h, \phi_h)$ describes the direction of the momentum π_1 in the ρ **helicity** frame. The ρ **decay amplitude** (a complex constant) has been absorbed into F_{λ}^J . w is the effective mass for the ρ and $\Delta(w)$ is the **Breit-Wigner form**

$$(90) \quad \Delta(w) = \frac{w_0 \Gamma_0}{w_0^2 - w^2 - i w_0 \Gamma(w)}, \quad \Gamma(w) = \Gamma_0 \left(\frac{w_0}{w} \right) \left(\frac{q}{q_0} \right) \left(\frac{q}{q_0} \right)^2$$

where q is the **breakup momentum** of the ρ in the ρ **RF**.

(b) Consider now the decay $b_1(1235) \rightarrow \omega\pi$.

Because of parity conservation, we must have $\ell = 0$ (**S** wave) or $\ell = 2$ (**D** wave). So we see that

$$(91) \quad F_{\lambda}^J = \sqrt{\frac{1}{3}} a_0^J + \sqrt{\frac{5}{3}} a_2^J (20 s\lambda | J\lambda), \quad J = s = 1$$

Evaluating the Clebsch-Gordan coefficients, we find

$$(92) \quad \left\{ \begin{array}{l} F_+^J = \sqrt{\frac{1}{3}} a_0^J + \sqrt{\frac{1}{6}} a_2^J \\ F_0^J = \sqrt{\frac{1}{3}} a_0^J - \sqrt{\frac{2}{3}} a_2^J \\ F_-^J = \sqrt{\frac{1}{3}} a_0^J + \sqrt{\frac{1}{6}} a_2^J \end{array} \right.$$

Note that $\sum_\lambda |F_\lambda^J|^2 = \sum_\ell |a_\ell^J|^2$ in which the interference terms cancel out.

Include the decay of the $\omega \rightarrow \pi_1 + \pi_2 + \pi_3$. Then the **overall** amplitude is

$$(93) \quad A^J(M; \Omega, \Omega_h) \propto \sum_\lambda F_\lambda^J D_{M\lambda}^{J*}(\phi, \theta, 0) D_{\lambda 0}^{s*}(\phi_h, \theta_h, 0), \quad J = s = 1, |\lambda| \leq 1$$

where $\Omega = (\theta, \phi)$ describes the direction of the momentum ω in the **JRF**, and $\Omega_h = (\theta_h, \phi_h)$ describes the direction of the momentum $\pi_1 \times \pi_2$ in the ω **helicity** frame. The ω **decay amplitude** can be integrated over

$$g^2 = \int |F_0^s(E_i)|^2 dE_1 dE_2, \quad s = 1 \text{ for } \omega$$

g has been **absorbed** into the F_λ^J .

Decay Modes involving γ 's in the final states:

(a) Consider the decays $\omega \rightarrow \gamma + \pi^0$ and $f_1(1285) \rightarrow \gamma + \rho^0$.

The relevant decay amplitude is

$$(94) \quad A_{\lambda_1 \lambda_2}^J(M; \Omega) \propto F_{\lambda_1 \lambda_2}^J D_{M\lambda}^{J*}(\phi, \theta, 0), \quad \lambda = \lambda_1 - \lambda_2, \quad \lambda_1 = \pm 1, \quad |\lambda| \leq J$$

Observe

$$\omega \rightarrow \gamma + \pi^0; \quad A_{\pm}^J(M; \Omega) \propto F_{\pm}^J D_{M\pm}^{J*}(\phi, \theta, 0); \quad F_{\pm}^J = -F_{\mp}^J, \quad J = 1$$

where $s_1 = 1$ and $\gamma_1 = \pm 1$ for γ . There is **one** non-zero helicity-coupling amplitude, i.e. F_{+}^J . And further note

$$f_1(1285) \rightarrow \gamma + \rho^0; \quad A_{\pm, \lambda_2}^J(M; \Omega) \propto F_{\pm, \lambda_2}^J D_{M\lambda}^{J*}(\phi, \theta, 0); \quad J = 1, \quad \lambda = \pm 1 - \lambda_2$$

where $s_1 = 1$ and $\gamma_1 = \pm 1$ for γ ; and $s_2 = 1$ and $\gamma_2 = \{-1, 0, +1\}$ for the ρ^0 . From parity conservation in the decay, one must have

$$(95) \quad F_{\pm, \lambda_2}^J = -F_{\mp, -\lambda_2}^J$$

There are **two** non-zero amplitudes, F_{++}^J and F_{+0}^J .

(b) Consider the decays $\pi^0 \rightarrow \gamma + \gamma$ or $a_0(980) \rightarrow \gamma + \gamma$.

Once again, we start with

$$(96) \quad A_{\lambda_1 \lambda_2}^J(M; \Omega) \propto F_{\lambda_1 \lambda_2}^J D_{M\lambda}^{J*}(\phi, \theta, 0), \quad \lambda = \lambda_1 - \lambda_2, \quad |\lambda| \leq J$$

where $s_1 = s_2 = 1$ and $\lambda_1 = \pm 1$ and $\lambda_2 = \pm 1$. From **parity conservation**, we have

$$(97) \quad F_{\lambda_1 \lambda_2}^J = \eta (-)^J F_{-\lambda_1 -\lambda_2}^J$$

By **Bose symmetry**, we should also have

$$(98) \quad F_{\lambda_1 \lambda_2}^J = (-)^J F_{\lambda_2 \lambda_1}^J$$

There is **one** non-zero element F_{++}^J for π^0 or $a_0(980)$.

Suppose $J = 1$. There is again **one** non-zero element F_{++}^J but $F_{++}^J = 0$ by Bose symmetry. So, **spin-one** particles cannot decay into two photons—**Landau-Yang Theorem**.

Consider now $J = 2$. There are **two** non-zero elements F_{++}^J and F_{+-}^J if $\eta = +1$, but there exist only **one** element F_{++}^J if $\eta = -1$.

Sequential Decays involving $\omega \rightarrow \gamma + \pi^0$

It is instructive to consider a problem investigated by the Crystal-Barrel collaboration.

$$(99) \quad \bar{p}p|_{\text{rest}}(^3S_1 \text{ or } ^1P_1) \rightarrow \omega + \pi^0(\pi_1) + \pi^0(\pi'_1) \quad \omega \rightarrow \gamma + \pi^0(\pi_2)$$

So we have $J = 1$ for the parent system. We fix the coordinate system in the decay plane, such that the z -axis is along the direction of ω and the y -axis is along the decay normal, $\hat{y} \propto \vec{\omega} \times \vec{\pi}_1$. We shall consider two intermediate states $b_1(1235) \rightarrow \omega + \pi_1$ and $b'_1(1235) \rightarrow \omega + \pi'_1$, so that we have $j = j' = s = s_1 = 1$. So there are **two decay chains** in this example:

$$[J]^M \xrightarrow{L} [j]^{\lambda_j} (\Omega_0) + \pi'_1, \quad [j]^{\lambda_j} \xrightarrow{\ell} [s]^\lambda (\Omega) + \pi_1, \quad [s]^\lambda \rightarrow [s_1]^{\lambda_1} (\Omega_2) + \pi_2$$

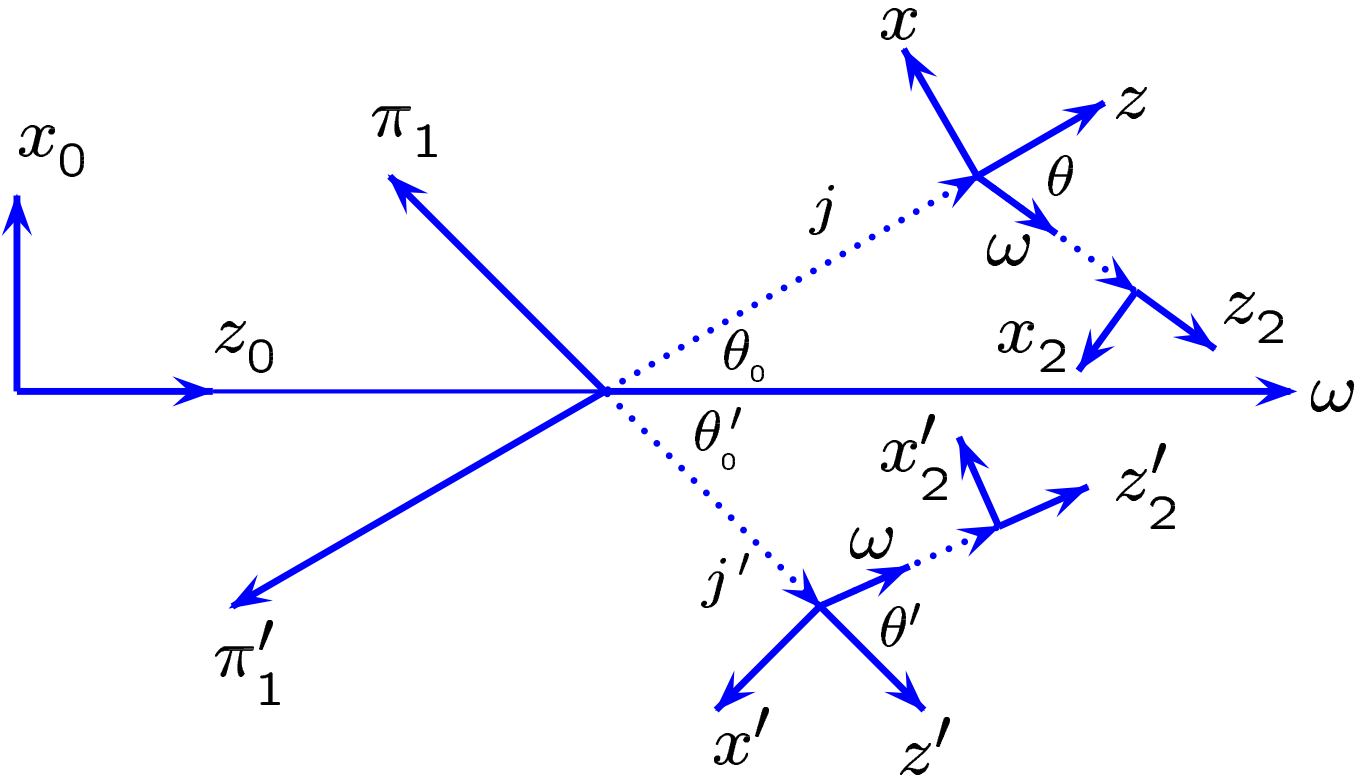
$$[J]^M \xrightarrow{L'} [j']^{\lambda'_j} (\Omega'_0) + \pi_1, \quad [j']^{\lambda'_j} \xrightarrow{\ell'} [s]^{\lambda'} (\Omega') + \pi'_1, \quad [s]^{\lambda'} \rightarrow [s_1]^{\lambda_1} (\Omega'_2) + \pi_2$$

where L (L') and ℓ (ℓ') are the orbital angular momenta in J and j (j') RFs, respectively. There is a **third type of isobar**:

$$[J]^M \xrightarrow{L_3} [l_3]^{\lambda_3} (\Omega_3) + [s]^\lambda, \quad [s]^\lambda \rightarrow [s_1]^{\lambda_1} (\Omega_1) + \pi_2$$

where $M = \lambda - \lambda_3$.

The processes are illustrated below.



The amplitudes for $J \rightarrow j(\Omega_0) + \pi'_1$ and $J \rightarrow j'(\Omega'_0) + \pi_1$ are

$$(100) \quad \begin{aligned} A_{M \lambda_j}^{J j}(\Omega_0) &\propto H_{\lambda_j}^J D_{M \lambda_j}^{J*}(0, \theta_0, 0) = H_{\lambda_j}^J d_{M \lambda_j}^J(\theta_0) \\ A_{M \lambda'_j}^{J j'}(\Omega'_0) &\propto \bar{H}_{\lambda'_j}^J D_{M \lambda'_j}^{J*}(\pi, \theta'_0, 0) = \exp[i M \pi] \bar{H}_{\lambda'_j}^J d_{M \lambda'_j}^J(\theta'_0) \end{aligned}$$

where

$$(101) \quad \begin{aligned} H_{\lambda_j}^J &= \sum_L \left(\frac{2L+1}{2j+1} \right)^{1/2} K_L^J(L 0 j \lambda_j | J \lambda_j) \\ \bar{H}_{\lambda'_j}^J &= \sum_{L'} \left(\frac{2L'+1}{2j'+1} \right)^{1/2} \bar{K}_{L'}^J(L' 0 j' \lambda'_j | J \lambda'_j) \end{aligned}$$

We allow for different states for j and j' ; for example, the j could stand for the $b_1(1235)$, while the j' might represent the $\rho(1700)$. The 'bar's over H^J and G^J indicate different amplitudes for these states. However, if both intermediate states happen to be the $b_1(1235)$, then the Bose symmetrization requires that $\bar{H}^J = H^J$ and $\bar{G}^J = G^J$.

We use the helicity description for the decay amplitude for $j \rightarrow s + \pi_1$

$$A_{\lambda_j \lambda}^j(\Omega) \propto F_\lambda^j D_{\lambda_j \lambda}^{j*}(\phi, \theta, 0)$$

(102)

$$F_\lambda^j = \sum_\ell \left(\frac{2\ell + 1}{2j + 1} \right)^{1/2} G_\ell^j(\ell 0 s \lambda | j \lambda)$$

where $\Omega = (\theta, \phi)$ describes the direction of s in the j RF (rest frame) [see Fig. above], and G_ℓ^j is the decay coupling constant for $j \rightarrow s + \pi_1$ with an orbital angular momentum ℓ . The decay amplitude for $j \rightarrow s + \pi_1$, followed by $s \rightarrow s_1 + \pi_2$, is

$$A_{\lambda_j \lambda_1}^j(\Omega, \Omega_2) \propto \sum_\lambda A_{\lambda_j \lambda}^j(\Omega) f_{\lambda_1}^s D_{\lambda \lambda_1}^{s*}(\phi_2, \theta_2, 0)$$

(103)

$$\propto \sum_\lambda F_\lambda^j D_{\lambda_j \lambda}^{j*}(\phi, \theta, 0) f_{\lambda_1}^s D_{\lambda \lambda_1}^{s*}(\phi_2, \theta_2, 0)$$

$f_{\lambda_1}^s$ is the helicity-coupling amplitude corresponding to $s \rightarrow s_1 + \pi_2$. For the example of $\omega \rightarrow \gamma + \pi^0$, we have $f_{\pm}^s = -f_{\mp}^s$ and $f_0^s = 0$. The angles $\Omega_2 = (\theta_2, \phi_2)$ describes the direction of s_1 in the s RF [see Fig. above].

The amplitude corresponding to the decay chain $j' \rightarrow s + \pi'_1$, followed by $s \rightarrow s_1 + \pi_2$, is

$$A_{\lambda'_j \lambda_1}^{j'}(\Omega', \Omega'_2) \propto \sum_{\lambda} F_{\lambda}^{j'} D_{\lambda'_j \lambda}^{j'*}(\phi', \theta', 0) f_{\lambda_1}^s D_{\lambda \lambda_1}^{s*}(\phi'_2, \theta'_2, 0)$$

(104)

$$F_{\lambda}^{j'} = \sum_{\ell'} \left(\frac{2\ell' + 1}{2j' + 1} \right)^{1/2} G_{\ell'}^{j'}(\ell' 0 s \lambda | j' \lambda)$$

The angles $\Omega' = (\theta', \phi')$ correspond to the direction s in the j' RF, while the angles $\Omega'_2 = (\theta'_1, \phi'_2)$ describe the direction of s_1 in the s RF. It is clear that the angles Ω_2 and Ω'_2 are different, because of the different paths taken to get to the s RF.

The overall amplitudes are, with $\Omega(\theta, \pi)$ and $\Omega'(\theta', \pi)$,

$$A_{M \lambda_1}^{J j}(\Omega_0, \Omega, \Omega_2) \propto \sum_{\lambda_j \lambda} A_{M \lambda_j}^{J j}(\Omega_0) A_{\lambda_j \lambda_1}^j(\Omega, R, \Omega_1)$$

$$\propto \sum_{\lambda_j} \exp[i \lambda_j \pi] H_{\lambda_j}^J d_{M \lambda_j}^J(\theta_0) \sum_{\lambda} F_{\lambda}^j d_{\lambda_j \lambda}^j(\theta) f_{\lambda_1}^s D_{\lambda \lambda_1}^{s*}(\phi_2, \theta_2, 0)$$

$$A_{M \lambda_1}^{J j'}(\Omega'_0, \Omega', \Omega'_2) \propto \sum_{\lambda'_j \lambda} A_{M \lambda'_j}^{J j'}(\Omega'_0) A_{\lambda'_j \lambda_1}^{j'}(\Omega', R', \Omega_1)$$

$$\propto \sum_{\lambda'_j} (-)^{M+\lambda'_j} \bar{H}_{\lambda'_j}^J d_{M \lambda'_j}^J(\theta'_0) \sum_{\lambda} F_{\lambda}^{j'} d_{\lambda'_j \lambda}^{j'}(\theta') f_{\lambda_1}^s D_{\lambda \lambda_1}^{s*}(\phi'_2, \theta'_2, 0)$$

Next we work out the third type of isobar, i.e. that of the dipion system $\pi_1 + \pi'_1$ described by $|\ell_3 \lambda_3\rangle$. The overall decay amplitude for $|JM\rangle \rightarrow |s\lambda\rangle + |\ell_3 \lambda_3\rangle$ is

$$(105) \quad A_{M \lambda_1}^{J \ell_3}(\Omega_1, \Omega_3) \propto \sum_{\lambda \lambda_3} E_{\lambda \lambda_3}^J D_{M \lambda - \lambda_3}^{J*}(0, 0, 0) D_{\lambda_3 0}^{\ell_3*}(\phi_3, \theta_3, 0) A_{\lambda \lambda_1}^s(\Omega_1)$$

Note that $M = \lambda - \lambda_3$. $E_{\lambda \lambda_3}^J$ is the usual helicity-coupling amplitude

$$(106) \quad E_{\lambda \lambda_3}^J = \sum_{L_3 S} \left(\frac{2L_3 + 1}{2J + 1} \right)^{1/2} G_{L_3 S}^J(L_3 0 SM | JM)(s \lambda \ell_3 - \lambda_3 | SM)$$

$\Omega_3(\theta_3, \phi_3)$ is measured in the dipion RF defined by $(-\hat{x}_0, \hat{y}_0, -\hat{z}_0)$. In general, $\phi_3 = 0$ or $\phi_3 = \pi$, but we can always set $\phi_3 = 0$ by allowing negative values of θ_3 , i.e. $-\pi < \theta_3 < \pi$. The overall amplitude becomes

$$(107) \quad A_{M \lambda_1}^{J \ell_3}(\Omega_3, \Omega_1) \propto \sum_{\lambda_3} E_{\lambda \lambda_3}^J d_{\lambda_3 0}^{\ell_3}(\theta_3) f_{\lambda_1}^s D_{\lambda \lambda_1}^{s*}(\phi_1, \theta_1, 0)$$

where $\lambda = M + \lambda_3$ and so there is no summation on λ .

The state $|s\lambda\rangle_2$ for ω in $(\hat{x}_2, \hat{y}_2, \hat{z}_2)$ can be described by the state $|s\nu\rangle_0$ given in $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ via

$$\begin{aligned}
 |s\lambda\rangle_2 &= R(\pi, \beta, 0) |s\lambda\rangle_0 = \sum_{\nu} |s\nu\rangle_0 \langle s\nu| R(\pi, \beta, 0) |s\lambda\rangle_0 = \sum_{\nu} D_{\nu\lambda}^s(\pi, \beta, 0) |s\nu\rangle_0 \\
 (108) \qquad &= \sum_{\nu} \exp[-i\nu\pi] d_{\nu\lambda}^s(\beta) |s\nu\rangle_0
 \end{aligned}$$

So the overall decay amplitude with respect to the state $|s\lambda\rangle_0$ is

$$(109) \qquad A_{M\lambda}^{Jj}(\Omega_0, \Omega) \propto \sum_{\lambda_j} H_{\lambda_j}^J d_{M\lambda_j}^J(\theta_0) F_{\lambda}^j d_{\lambda_j\lambda}^j(\theta) \sum_{\nu} (-)^{\lambda_j+\nu} d_{\nu\lambda}^s(\beta)$$

Note that n of the sign factor $(-)^n$ given above is an integer, but that we have $n = \lambda_j + \nu$ and *not* $n = \lambda_j - \nu$, because ω is in the final state and so we must use $\langle s\lambda|$.

Consider next the decay amplitude for $J \rightarrow j' + \pi - 1$ with $j' \rightarrow s + \pi'_1$.
 The $|s\lambda\rangle'_2$ for ω in $(\hat{x}'_2, \hat{y}'_2, \hat{z}'_2)$ can be expressed by

$$(110) \quad |s\lambda\rangle'_2 = R(0, \beta', 0) |s\lambda\rangle_0 = \sum_{\nu} d_{\nu\lambda}^s(\beta') |s\nu\rangle_0$$

The overall decay amplitude with respect to the state $|s\lambda\rangle_0$ is

$$(111) \quad A_{M\lambda}^{Jj'}(\Omega_0, \Omega') \propto \sum_{\lambda'_j} (-)^{M+\lambda'_j} \bar{H}_{\lambda'_j}^J d_{M\lambda'_j}^J(\theta'_0) \\ \sum_{\lambda} F_{\lambda}^{j'} d_{\lambda'_j\lambda}^{j'}(\theta') \sum_{\nu} d_{\nu\lambda}^s(\beta')$$

The $|s\nu\rangle_0$ decay, i.e. $\omega \rightarrow \gamma + \pi^0$, is

$$(112) \quad \langle \Omega_1, \lambda_1 | \mathcal{M}_s | s\nu \rangle_0 = f_{\lambda_1}^s D_{\nu\lambda_1}^{s*} \phi_1, \theta_1, 0)$$

Let V_{Jj} , $V_{Jj'}$ and $V_{J\ell_3}$ be the parameters (complex in general) which govern the strength of each isobar. The parameters should be a function of J but *not* of either M or the photon helicity λ_1 . We see that

$$\begin{aligned}
 A_{M\lambda_1}^J = & V_{Jj} \left\{ \sum_{\lambda_j} H_{\lambda_j}^J d_{M\lambda_j}^J(\theta_0) \sum_{\lambda} F_{\lambda}^j d_{\lambda_j\lambda}^j(\theta) \right. \\
 & \left. \times \sum_{\nu} (-)^{\lambda_j+\nu} d_{\nu\lambda}^s(\beta) \right\} f_{\lambda_1}^s D_{\nu\lambda_1}^{s*}(\phi_1, \theta_1, 0) \\
 + & V_{Jj'} \left\{ \sum_{\lambda'_j} (-)^{M+\lambda'_j} \bar{H}_{\lambda'_j}^J d_{M\lambda'_j}^J(\theta'_0) \sum_{\lambda} F_{\lambda}^{j'} d_{\lambda'_j\lambda}^{j'}(\theta') \right. \\
 & \left. \times \sum_{\nu} d_{\nu\lambda}^s(\beta') \right\} f_{\lambda_1}^s D_{\nu\lambda_1}^{s*}(\phi_1, \theta_1, 0) \\
 + & V_{J\ell_3} \left\{ \sum_{\lambda_3} E_{\nu\lambda_3}^J d_{\lambda_3 0}^{\ell_3}(\theta_3) \right\} f_{\lambda_1}^s D_{\nu\lambda_1}^{s*}(\phi_1, \theta_1, 0)
 \end{aligned}$$

where $\nu = M + \lambda_3$. The decay amplitude for $\omega \rightarrow \gamma + \pi^0$ can now be factored out in the expression given above.

The result can be applied to an $N\pi\pi$ system with very little modification.

$$\begin{aligned}
 A_{M\lambda}^J = & V_{Jj} \left\{ \sum_{\lambda_j} H_{\lambda_j}^J d_{M\lambda_j}^J(\theta_0) F_{\lambda}^j d_{\lambda_j\lambda}^j(\theta) \sum_{\nu} (-)^{\lambda_j+\nu} d_{\nu\lambda}^s(\beta) \right\} \\
 & + V_{Jj'} \left\{ \sum_{\lambda'_j} (-)^{M+\lambda'_j} \bar{H}_{\lambda'_j}^J d_{M\lambda'_j}^J(\theta'_0) F_{\lambda}^{j'} d_{\lambda'_j\lambda}^{j'}(\theta') \sum_{\nu} d_{\nu\lambda}^s(\beta') \right\} \\
 & + V_{J\ell_3} \left\{ E_{\lambda\lambda_3}^J d_{\lambda_3 0}^{\ell_3}(\theta_3) \right\}
 \end{aligned}$$

where $\lambda_3 = \lambda - M$.

Canonical prescription for decay amplitudes:

For $J \rightarrow j + \pi'_1$ we find

$$(113) \quad \begin{aligned} A_{M m_j}^{J j}(\Omega_0) &\propto \sum_L K_L^J(L M_L j m_j | J M) D_{M_L 0}^{L*}(\phi_0, \theta_0, 0) \\ &\propto \sum_L K_L^J(L M_L j m_j | J M) d_{M_L 0}^L(\theta_0) \end{aligned}$$

with $\phi_0 = 0$ and $M_L = M - m_j$. For $j \rightarrow s + \pi_1$ we have

$$(114) \quad A_{m_j \lambda}^j(\Omega_c) \propto (-)^{m_j - \lambda} \sum_{\ell} G_{\ell}^j(\ell m s \lambda | j m_j) d_{m 0}^{\ell}(\theta_c), \quad \theta_c > 0$$

where $\phi_c = 0$. The overall amplitude is

$$(115) \quad \begin{aligned} A_{M \lambda}^{J j}(\Omega_0, \Omega_c) &\propto \sum_{L m_j} K_L^J(L M_L j m_j | J M) \\ &\quad \times (-)^{m_j - \lambda} \sum_{\ell} G_{\ell}^j(\ell m s \lambda | j m_j) d_{M_L 0}^L(\theta_0) d_{m 0}^{\ell}(\theta_c) \end{aligned}$$

A concrete example:

$$\bar{p}p \rightarrow \omega\pi_1\pi'_1$$

In the overall CM system, let

$$(\hat{x}_0, \hat{y}_0, \hat{z}_0) : \hat{x}_0 = (1, 0, 0)$$

$$\hat{y}_0 = (0, 1, 0)$$

$$\hat{z}_0 = (0, 0, 1)$$

4-mom	E	p_x	p_y	p_z
$\bar{p}p$	1.7699	0	0	0
ω	0.9857	0	0	0.6000
π_1	0.3156	0.2000	0	-0.2000
π'_1	0.4686	-0.2000	0	-0.4000

Relevant Angles:

$$\theta_0 = +26.5650^\circ,$$

$$\theta'_0 = -45.000^\circ$$

$$\theta = -51.8542^\circ,$$

$$\theta' = -60.8024^\circ$$

In the ω RF, let

4-mom	E	p	θ	ϕ
ω	0.7820	0	0	0
γ	0.3785	0.3785	30°	60°
π^0	0.4035	0.3785	150°	240°

$$\hat{x}_2 = (-0.9042, 0, -0.4272), \quad \hat{y}_2 = (0, -1, 0), \quad \hat{z}_2 = (-0.4272, 0, +0.9042)$$

$$\hat{x}'_2 = (+0.9622, 0, -0.2723), \quad \hat{y}'_2 = (0, +1, 0), \quad \hat{z}'_2 = (+0.2723, 0, +0.9622)$$

γ direction	θ	ϕ
Ω_2 in $(\hat{x}_2, \hat{y}_2, \hat{z}_2)$	44.8459°	-142.1188°
Ω'_2 in $(\hat{x}'_2, \hat{y}'_2, \hat{z}'_2)$	25.8290°	83.6494°

Wigner rotations:

$$\beta = +25.2891^\circ, \quad \beta' = +15.8024^\circ$$

Density Matrix

Here we derive the **symmetry properties** of the general spin-density matrix for a **system c** produced in

$$(116) \quad a + b \rightarrow c + d$$

where the participating particles are arbitrary and include **mesons, baryons as well as photons**. We assume that c is an **intermediate** state, mesonic or baryonic, and it couples to any number of allowed decay channels.

The density matrices in the **reflectivity basis** were given by

S. U. Chung and T. L. Trueman, *Phys. Rev. D* **11**, 633 (1975).

For a deeper understanding of the quantum treatment of one-particle states, the reader is referred to the book by

Steven Weinberg, '*The Quantum Theory of Fields*,' Volume I
(Cambridge University Press, Cambridge, 1995), Chapter 2.

General Angular Distributions in Reflectivity Basis:

Let $|jm\rangle$ be the **spin state**^a for c where the quantization axis is defined in the **production plane**, i.e. one takes either the **helicity** or the **Jackson** frame for c . The amplitude for production and decay of the c is

$$(117) \quad A \propto \sum_{\chi m} \langle \vec{p}_c \chi m, \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle D_m^\chi(\tau)$$

where χ specifies the complete quantum state for c , which includes its **spin j** , its **parity**, its **C -parity**, its **isotopic spin**, and its **decay products** with the **phase-space element given by τ** ; λ 's refer to the helicities; T is the transition operator of the process $ab \rightarrow cd$; and D is the **decay amplitude for c** , which may consist of a product of the '**rotation functions**' as well as the **Breit-Wigner forms**. The distribution function follows immediately

$$(118) \quad I(\tau) \propto \sum_{\lambda_a \lambda_b \lambda_d} \sum_{\chi m \chi' m'} \times \langle \vec{p}_c \chi m, \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c \chi' m', \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle^* \times D_m^\chi(\tau) D_{m'}^{\chi'}{}^*(\tau)$$

^a We use j to denote the spin and NOT J as used in the previous sections.

Note that the helicities of a , b and d are the 'external' unobserved variables and therefore summed over outside of the absolute square of the amplitude A . In terms of the generalized spin-density matrix

$$(119) \quad \rho_{mm'}^{\chi\chi'} \propto \sum_{\lambda_a \lambda_b \lambda_d} \langle \vec{p}_c \chi m, \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle \langle \vec{p}_c \chi' m', \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle^*$$

the distribution function assumes an elegant form

$$(120) \quad I(\tau) \propto \sum_{\substack{\chi m \\ \chi' m'}} \rho_{mm'}^{\chi\chi'} D_m^\chi(\tau) D_{m'}^{\chi'}{}^*(\tau)$$

We are now ready to introduce the **reflection operator** through the production plane for $ab \rightarrow cd$. Let this plane be defined to be the x - z plane, i.e. **the production normal is along the y -axis**. Then the reflection operator defined by

$$(121) \quad \Pi_y = U[R_y(\pi)] \Pi = \Pi U[R_y(\pi)]$$

where $U[R_y(\pi)]$ is a unitary operator representing a rotation by π around the y -axis, i.e.

$$(122) \quad U[R_y(\pi)] = \exp(-i\pi J_y)$$

which is simply given by the standard d -function

$$(123) \quad U_{m' m}[R_y(\pi)] = d_{m' m}^j(\pi) = (-)^{j-m} \delta_{m', -m}$$

Note also that

$$(124) \quad U_{m' m}[R_y(-\pi)] = d_{m' m}^j(-\pi) = (-)^{j+m} \delta_{m', -m}$$

Let Λ is a general operator in the xz -plane. Then, we see that

$$(125) \quad \left[\Pi_y, U[\Lambda(\vec{p}_c)] \right] = 0, \quad \Pi_y |\vec{p}_c \chi m\rangle = \eta_c (-)^{j-m} |\vec{p}_c \chi -m\rangle, \quad \Pi_y^2 = (-)^{2j} I$$

where η_c is the intrinsic parity of the c . Also **true** for **helicity** states ($|\vec{p}_i \lambda_i\rangle$, $i = a, b, d$). Also **true** for **massless** particles ($m \rightarrow \lambda = \pm j$).

We move over to the reflection-basis states for c , i.e.

$$(126) \quad |\vec{p}_c \epsilon \chi m\rangle = \theta(m) \{ |\vec{p}_c \chi m\rangle + \epsilon \eta_c (-)^{j-m} |\vec{p}_c \chi -m\rangle \}$$

where

$$(127) \quad \theta(m) = \frac{1}{\sqrt{2}}, m > 0; \quad \theta(m) = \frac{1}{2}, m = 0; \quad \theta(m) = 0, m < 0$$

These basis states constitute eigenvectors of the reflection operator

$$(128) \quad \epsilon^2 = (-)^{2j} \rightarrow \Pi_y |\vec{p}_c \epsilon \chi m\rangle = \epsilon (-)^{2j} |\vec{p}_c \epsilon \chi m\rangle$$

so that we have $\epsilon = \pm 1$ for bosons and $\epsilon = \pm i$ for fermions. Note, in addition, that $\epsilon \epsilon^* = |\epsilon|^2 = 1$ for both bosons and fermions.

The generalized density matrix in the reflectivity basis is, with $m \geq 0$ and $m' \geq 0$,

$$(129) \quad \epsilon \epsilon' \rho_{mm'}^{\chi \chi'} \propto \sum_{\lambda_a \lambda_b \lambda_d} \langle \epsilon \vec{p}_c \chi m, \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle \langle \epsilon' \vec{p}_c \chi' m', \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle^*$$

We shall explore the consequences of a reflection operation applied to the transition matrices above. The key observation is that Π_y leaves the transition operator T unperturbed, i.e. $[\Pi_y, T] = 0$. The three-vectors \vec{p}_i ($i = a, b, c, d$) are left unchanged under Π_y , i.e. the boost operators and/or the rotations about the y -axis, which enter in the definitions of the helicity or the Jackson frames, remain invariant under Π_y . Therefore, the parity conservation is relegated to exploring the consequences of Π_y acting on the 'rest' states. Insert $\Pi_y^{-1} \Pi_y = \Pi_y^\dagger \Pi_y = I$ next to each T and propagate Π_y^\dagger and Π_y backwards and forwards, respectively, to find

$$(130) \quad \epsilon \epsilon' \rho_{mm'}^{\chi \chi'} \propto \epsilon \epsilon'^* (-)^{2(j-j')} \sum_{\lambda_a \lambda_b \lambda_d}$$

$$\langle \epsilon \vec{p}_c \chi m, \vec{p}_d, -\lambda_d | T | \vec{p}_a, -\lambda_a, \vec{p}_b, -\lambda_b \rangle \langle \epsilon' \vec{p}_c \chi' m', \vec{p}_d, -\lambda_d | T | \vec{p}_a, -\lambda_a, \vec{p}_b, -\lambda_b \rangle^*$$

so that

$$(131) \quad \epsilon \epsilon' \rho_{mm'}^{\chi \chi'} = \epsilon \epsilon'^* \times \epsilon \epsilon' \rho_{mm'}^{\chi \chi'}$$

So we see that $\epsilon \epsilon'^* = +1$. Multiply it by ϵ' from the right and noting that $\epsilon' \epsilon'^* = |\epsilon'|^2 = +1$, we find $\epsilon = \epsilon'$. Here we have carefully handled the derivation, so that the formula above applies to both bosons and fermions.

The density matrix can be written, quite generally, in a **block-diagonal** form

$$(132) \quad \epsilon \rho_{mm'}^{\chi \chi'} \propto \sum_{\lambda_a \lambda_b \lambda_d} \langle \epsilon \vec{p}_c \chi m, \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle \langle \epsilon \vec{p}_c \chi' m', \vec{p}_d \lambda_d | T | \vec{p}_a \lambda_a, \vec{p}_b \lambda_b \rangle^*$$

a **fundamental formula** which incorporates **parity conservation** in the production process $ab \rightarrow cd$. The distribution function in the reflectivity basis is

$$(133) \quad I(\tau) \propto \sum_{\epsilon}^2 \sum_{\substack{\chi m \\ \chi' m'}} \epsilon \rho_{mm'}^{\chi \chi'} \epsilon D_m^{\chi}(\tau) \epsilon D_{m'}^{\chi'}{}^*(\tau), \quad m \geq 0, m' \geq 0$$

where ϵD is the decay amplitude in the reflectivity basis. Consider a simple decay

$$c \rightarrow s_1(\lambda_1) + s_2(\lambda_2)$$

In the **cRF**, we have, with $\vec{p}_c = 0$ and $\tau = R(\phi, \theta, 0)$,

$$(134) \quad \begin{aligned} \epsilon D_m^{\chi}(\tau) &= \langle \vec{q} \lambda_1; -\vec{q} \lambda_2 | \mathcal{M} | \epsilon j m \rangle \\ &= N_j F_{\lambda_1 \lambda_2}^j \theta(m) \left\{ D_{m\lambda}^{j*}(R) + \epsilon \eta_c (-)^{j-m} D_{-m\lambda}^{j*}(R) \right\}, \quad \lambda = \lambda_1 - \lambda_2 \end{aligned}$$

The **rank of the density matrix** is determined by the number of independent terms in the summation on helicities. Let

$$n_i = 2s_i + 1 \quad \text{or} \quad n_i = 2 \text{ (photons)} \quad \text{for} \quad i = a, b, d$$

depending on whether a particle is massive or massless. The total number in the sum is

$$(135) \quad N = n_a n_b n_d$$

So the **rank** of the density matrix is $(N + 1)/2$ if N is **odd**, and it is $N/2$ if N is **even**. Note that the reduction in the rank comes from parity conservation in the production process (to show this, **apply Π_y** again to the amplitudes).

Table I. Rank of Spin-Density Matrix for X

Reaction	Rank
$\pi^- p \rightarrow \pi^- X^+$	1
$\pi^- p \rightarrow X^- p$	2
$\pi^+ n \rightarrow X^- \Delta^{++}$	4
$\bar{p} p \rightarrow X^- p$	4
$\bar{n} p \rightarrow X^- \Delta^{++}$	8
$\gamma p \rightarrow X^0 p$	4
$\gamma p \rightarrow \pi^+ X^0$	2
$\nu p \rightarrow e^- X^{++}$	1 [†]
$e^- p \rightarrow e^- X^+$	1 [†]
$\phi p \rightarrow X^0 p$	6
$\phi p \rightarrow \pi^+ X^0$	3
$\pi^- \eta \rightarrow \pi^- X^0$	1
$\pi^- \phi \rightarrow \pi^- X^0$	2

[†] The electrons are assumed to come with one helicity.

True in general for **odd-half-integer spins** in the limit of zero mass.

Consider an $N_\epsilon \times N_\epsilon$ density matrix, with $i = \{\chi m\}$ and $j = \{\chi' m'\}$,

$$(136) \quad {}^\epsilon \rho_{ij} = \sum_{k=1}^{K_\epsilon} {}^\epsilon V_{ik} {}^\epsilon V_{jk}^*, \quad \Rightarrow \quad {}^\epsilon \rho = {}^\epsilon V {}^\epsilon V^\dagger, \quad \Rightarrow \quad {}^\epsilon \rho = {}^\epsilon \rho^\dagger$$

where $i, j = 1, \dots, N_\epsilon$; $k = 1, \dots, K_\epsilon$, and $K_\epsilon (= 1, \dots, \infty)$ is the **rank** of the density matrix. Note that ${}^\epsilon \rho$ is an $N_\epsilon \times N_\epsilon$ **square** matrix, whereas ${}^\epsilon V$ is, in general, a **retangular** matrix $N_\epsilon \times K_\epsilon$. The '**Cholesky**' decomposition of ${}^\epsilon V$ is, e.g. for $N_\epsilon = 6$ or for 6×6 ${}^\epsilon \rho$,

$$(137) \quad {}^\epsilon V = \{{}^\epsilon V_{ik}\} = \begin{pmatrix} {}^\epsilon V_{11} & 0 & 0 & 0 & 0 & 0 \\ {}^\epsilon V_{21} & {}^\epsilon V_{22} & 0 & 0 & 0 & 0 \\ {}^\epsilon V_{31} & {}^\epsilon V_{32} & {}^\epsilon V_{33} & 0 & 0 & 0 \\ {}^\epsilon V_{41} & {}^\epsilon V_{42} & {}^\epsilon V_{43} & {}^\epsilon V_{44} & 0 & 0 \\ {}^\epsilon V_{51} & {}^\epsilon V_{52} & {}^\epsilon V_{53} & {}^\epsilon V_{54} & {}^\epsilon V_{55} & 0 \\ {}^\epsilon V_{61} & {}^\epsilon V_{62} & {}^\epsilon V_{63} & {}^\epsilon V_{64} & {}^\epsilon V_{65} & {}^\epsilon V_{66} \end{pmatrix}$$

where ${}^\epsilon V_{ik}$ is complex in general but ${}^\epsilon V_{ii} = \text{real} \geq 0$. There are 6 real diagonal elements and 15 complex off-diagonal elements of ${}^\epsilon V$, for a total of **36 parameters** required to describe a 6×6 ${}^\epsilon \rho$. The **rank** is given by the **number of columns** counting from the left, with the rest being zero. For example, if the **rank=2**, then we must have ${}^\epsilon V_{ik} = 0, i \geq k \geq 3$. In this case, there are 2 real diagonal elements and 9 complex off-diagonal elements, for a total of **20 parameters** in the problem.

Now go back to the notation $\{\chi m\}$ and $\{\chi' m'\}$

$$(138) \quad \epsilon \rho_{mm'}^{\chi \chi'} = \sum_{k=1}^{K_\epsilon} \epsilon V_{mk}^\chi \epsilon V_{m'k}^{\chi' *} \quad , \quad m \geq 0, m' \geq 0$$

and define

$$(139) \quad \epsilon U_k(\tau) = \sum_{\chi m}^{N_\epsilon} \epsilon V_{mk}^\chi \epsilon D_m^\chi(\tau), \quad m \geq 0$$

The distribution function

$$I(\tau) \propto \sum_{\epsilon}^2 \sum_{\substack{\chi m \\ \chi' m'}}^{N_\epsilon} \epsilon \rho_{mm'}^{\chi \chi'} \epsilon D_m^\chi(\tau) \epsilon D_{m'}^{\chi' *}(\tau)$$

becomes

$$(140) \quad I(\tau) \propto \sum_{\epsilon}^2 \sum_{k=1}^{K_\epsilon} \left| \epsilon U_k(\tau) \right|^2$$

Maximum-Likelihood Method

Introduce the so-called **extended likelihood function** for finding 'n' events in a given mass bin

$$(141) \quad \mathcal{L} \propto \left[\frac{\bar{n}^n}{n!} e^{-\bar{n}} \right] \prod_i^n \left[\frac{I(\tau_i)}{\int I(\tau) \eta(\tau) \phi(\tau) d\tau} \right]$$

where $\eta(\tau)$ is the **experimental finite acceptance** at τ and the invariant phase-space element given by

$$(142) \quad d\phi = \left(\frac{d\phi}{d\tau} \right) d\tau = \phi(\tau) d\tau$$

The first bracket in \mathcal{L} represents the **Poisson probability** for finding 'n' events in the mass bin, and the expectation value \bar{n} is

$$(143) \quad \bar{n} \propto \int I(\tau) \eta(\tau) \phi(\tau) d\tau$$

The likelihood function \mathcal{L} can now be written, dropping the factors depending on n alone,

$$\mathcal{L} \propto \left[\prod_i^n I(\tau_i) \right] \exp \left[- \int I(\tau) \eta(\tau) \phi(\tau) d\tau \right]$$

The 'log' of the likelihood function now has the form,

$$(144) \quad \ln \mathcal{L} = \sum_i^n \ln I(\tau_i) - \int I(\tau) \eta(\tau) \phi(\tau) d\tau$$

We shall adopt the following shorthand notation

$$(145) \quad \alpha = \{\epsilon k; \chi m\} \quad \text{and} \quad \alpha' = \{\epsilon k; \chi' m'\}$$

and write

$$(146) \quad I(\tau) = \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* D_\alpha(\tau) D_{\alpha'}^*(\tau)$$

The so-called 'experimental' normalization integral is given by

$$(147) \quad \Psi_{\alpha\alpha'}^x = \int [D_\alpha(\tau) D_{\alpha'}^*(\tau)] \eta(\tau) \phi(\tau) d\tau$$

so that

$$(148) \quad \ln \mathcal{L} = \sum_i^n \ln \left[\sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* D_\alpha(\tau_i) D_{\alpha'}^*(\tau_i) \right] - \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \Psi_{\alpha\alpha'}^x$$

We explore the normalization for V 's by setting $V = x W$,
 where x is **independent** of α ,

$$(149) \quad \ln \mathcal{L} = \sum_i^n \ln \left[x^2 \sum_{\alpha\alpha'} W_\alpha W_{\alpha'}^* D_\alpha(\tau_i) D_{\alpha'}^*(\tau_i) \right] - x^2 \sum_{\alpha\alpha'} W_\alpha W_{\alpha'}^* \Psi_{\alpha\alpha'}^x$$

At the **maximum**, we should have

$$(150) \quad 0 = \frac{\partial \ln \mathcal{L}}{\partial x^2} = \sum_i^n \left[\frac{1}{x^2} \right] - \sum_{\alpha\alpha'} W_\alpha W_{\alpha'}^* \Psi_{\alpha\alpha'}^x$$

so that

$$(151) \quad \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \Psi_{\alpha\alpha'}^x = n$$

We can define the **theoretical** normalization integral, with $\eta(\tau) = 1$,

$$(152) \quad \Psi_{\alpha\alpha'} = \int \left[D_\alpha(\tau) D_{\alpha'}^*(\tau) \right] \phi(\tau) d\tau$$

The **predicted** number of events is

$$(153) \quad \begin{aligned} N &= \sum_{\alpha\alpha'} V_\alpha V_{\alpha'}^* \Psi_{\alpha\alpha'} \\ &\equiv \sum_{\alpha\alpha'} N_{\alpha\alpha'} , \quad N_{\alpha\alpha'} = V_\alpha V_{\alpha'}^* \Psi_{\alpha\alpha'} \end{aligned}$$

So the **predicted** number of events for a partial wave α is

$$(154) \quad N_{\alpha\alpha} = |V_\alpha|^2 \Psi_{\alpha\alpha} , \quad \Psi_{\alpha\alpha} = \int |D_\alpha(\tau)|^2 \phi(\tau) d\tau$$

The **predicted** number of events for the interference between the partial waves α and α' is

$$(155) \quad N_{\alpha\alpha'} + N_{\alpha'\alpha} = 2\Re \left\{ V_\alpha V_{\alpha'}^* \Psi_{\alpha\alpha'} \right\} , \quad \alpha \neq \alpha'$$

