

Selected Topics in Hadron Spectroscopy

Mathematical Techniques

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Introduction

- General References
- Spin Formalisms
- The Poincaré Group
- One-, Two- and Three-Particle States
- Treatment of Massless Particles
- Two- and Three-Body Decays
- Parallel Sequential Decays
- Reflectivity Operations
- Spin Density Matrices
- Techniques of Partial-Wave Analysis—
Extended Maximum-Likelihood Methods
- Flavor $SU(3)$
- Flux-Tube Model for Exotic Mesons

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Spin Formalisms

One-Particle States at Rest

States of a single particle at rest (mass $w > 0$) may be denoted by $|jm\rangle$, where j is the spin and m the z -component of the spin. The states $|jm\rangle$ are the canonical basis vectors by which the angular momentum operators are represented in the standard way. Since the angular momentum operators are the infinitesimal generators of the rotation operator, the spin of a particle characterizes how the particle at rest transforms under spatial rotations.

Let us denote the three components of the angular momentum operator by J_x , J_y , and J_z (or J_1 , J_2 , and J_3). They are Hermitian operators satisfying the following commutation relations:

$$(1) \quad [J_i, J_j] = i \epsilon_{ijk} J_k ,$$

where i , j , and k run from 1 to 3. The operators J_i act on the canonical basis vectors $|jm\rangle$ as follows:

$$(2) \quad \begin{aligned} J^2 |jm\rangle &= j(j+1) |jm\rangle \\ J_z |jm\rangle &= m |jm\rangle \\ J_{\pm} |jm\rangle &= [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}} |jm \pm 1\rangle , \end{aligned}$$

where $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_{\pm} = J_x \pm iJ_y$.

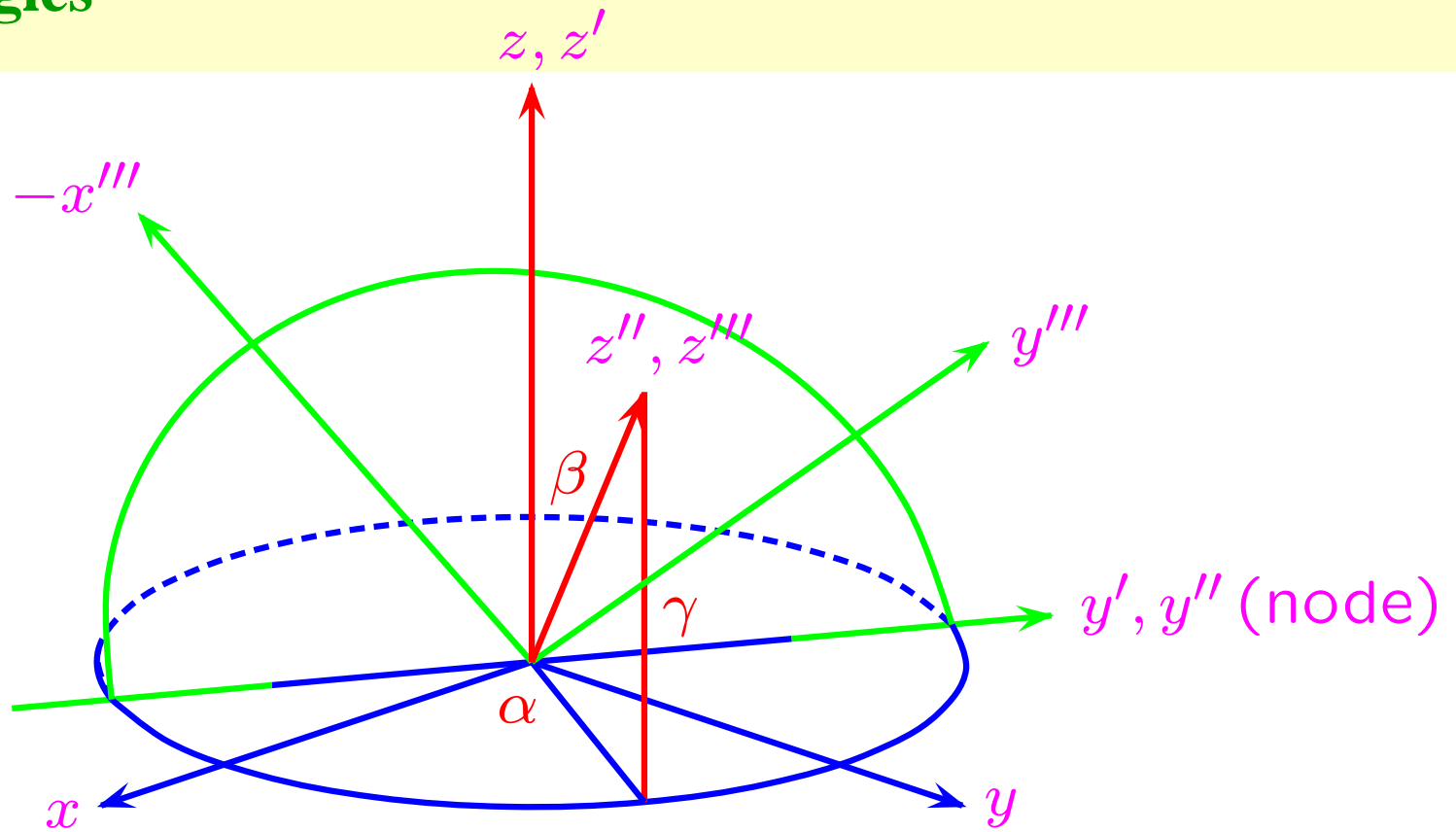
The states $|jm\rangle$ are normalized in the standard way and satisfy the completeness relation:

$$(3) \quad \langle j'm'|jm\rangle = \delta_{j'j} \delta_{m'm}, \quad \sum_{jm} |jm\rangle\langle jm| = I,$$

where I denotes the identity operator.

A finite rotation of a physical system (with respect to a fixed coordinate axis) may be denoted by $R(\alpha, \beta, \gamma)$, where (α, β, γ) are the standard **Euler angles**. We use the so-called **active rotations**.

Euler Angles



$$\begin{aligned}
 U[R(\alpha, \beta, \gamma)] &= \exp[-i \alpha J_z] \exp[-i \beta J_{y'}] \exp[-i \gamma J_{z''}] \\
 &= \exp[-i \gamma J_{z''}] \exp[-i \beta J_{y'}] \exp[-i \alpha J_z]
 \end{aligned}$$

$$D_{m'm}^j(\alpha, \beta, \gamma) = \langle jm' | U[R(\alpha, \beta, \gamma)] | jm \rangle$$

M. E. Rose,
 'Elementary Theory of Angular Momentum,'
 John Wiley & Sons, Inc. (see Chapter IV)

To each R , there corresponds a **unitary operator** $U[R]$, which acts on the states $|jm\rangle$, and preserves the multiplication law:

$$U[R_2 R_1] = U[R_2] U[R_1] .$$

Now the unitary operator representing a rotation $R(\alpha, \beta, \gamma)$ may be written

$$(4) \quad U[R(\alpha, \beta, \gamma)] = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$$

corresponding to the rotation of a physical system (**active rotation!**) by γ around the z -axis, β around the y -axis, and finally by α around the z -axis, with respect to a fixed (x, y, z) coordinate system. Then rotation of a state $|jm\rangle$ is given by

$$(5) \quad U[R(\alpha, \beta, \gamma)] |jm\rangle = \sum_{m'} |jm'\rangle D_{m'm}^j(\alpha, \beta, \gamma) ,$$

$D_{m'm}^j(\alpha, \beta, \gamma)$ is the matrix **representation** of the **rotation** group.

$D_{m'm}^j$ is the standard rotation matrix as given by **M. E. Rose**:

$$(6) \quad \begin{aligned} D_{m'm}^j(R) &= D_{m'm}^j(\alpha, \beta, \gamma) = \langle jm' | U[R(\alpha, \beta, \gamma)] | jm \rangle \\ &= e^{-i m' \alpha} d_{m'm}^j(\beta) e^{-i m \gamma} \end{aligned}$$

and

$$(7) \quad d_{m'm}^j(\beta) = \langle jm' | e^{-i \beta J_y} | jm \rangle .$$

In **Appendix A** of the “Spin Formalisms” (**S. U. Chung**), some useful formulae involving $D_{m'm}^j$ and $d_{m'm}^j$ are listed.

Relativistic One-Particle States

Relativistic one-particle states with momentum \vec{p} may be obtained by applying on the states $|jm\rangle$ a unitary operator which represents a Lorentz transformation that takes a particle at rest to a particle of momentum \vec{p} . There are two distinct ways of doing this, leading to **canonical** and **helicity** descriptions of relativistic free particle states. Let us first consider an arbitrary four-momentum p^μ defined by

$$(8) \quad p^\mu = (p^0, p^1, p^2, p^3) = (E, p_x, p_y, p_z) = (E, \vec{p}) .$$

With the metric tensor given by

$$(9) \quad g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

we can also define a four-momentum with lower indices:

$$(10) \quad p_\mu = g_{\mu\nu} p^\nu = (E, -\vec{p}) .$$

My convention: p is used to denote both the **four-momentum** and the magnitude of the **three-momentum**.

The proper homogeneous orthochronous Lorentz transformation takes the four-momentum p^μ into p'^μ as follows:

$$(11) \quad p'^\mu = \Lambda^\mu{}_\nu p^\nu ,$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix defined by

$$(12) \quad g_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = g_{\mu\nu}, \quad \det \Lambda = 1, \quad \Lambda^0{}_0 > 0 .$$

The Lorentz transformation given by $\Lambda^\mu{}_\nu$ includes, in general, rotations as well as the pure Lorentz transformations. Let us denote by $L^\mu{}_\nu(\vec{\beta})$ a pure time-like Lorentz transformation, where $\vec{\beta}$ is the velocity of the transformation. Of particular importance is the pure Lorentz transformation along the z -axis, denoted by $L_z(\beta)$:

$$(13) \quad L_z(\beta) = \begin{pmatrix} \cosh \alpha & 0 & 0 & \sinh \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \alpha & 0 & 0 & \cosh \alpha \end{pmatrix}$$

where $\beta = \tanh \alpha$.

In terms of $L_z(\beta)$, it is easy to define a pure Lorentz transformation along an arbitrary direction $\vec{\beta}$:

$$(14) \quad L(\vec{\beta}) = R(\phi, \theta, 0)L_z(\beta)R^{-1}(\phi, \theta, 0) ,$$

where $R(\phi, \theta, 0)$ is the rotation which takes the z -axis into the direction of $\vec{\beta}$ with spherical angles (θ, ϕ) :

$$(15) \quad \hat{\beta} = R(\phi, \theta, 0)\hat{z} .$$

The relation $L(\vec{\beta})$ is an obvious one, but the reader can easily check for a special case with $\phi = 0$:

$$(16) \quad R(\phi, \theta, 0) = \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix} , \quad R_{ij} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

Now the action of an arbitrary Lorentz transformation Λ on relativistic particle states may be represented by a unitary operator $U[\Lambda]$. The operator preserves the multiplication law, called the group property:

$$(17) \quad U[\Lambda_2 \Lambda_1] = U[\Lambda_2] U[\Lambda_1] .$$

Let us denote by $L(\vec{p})$ the “boost” which takes a particle with mass $w > 0$ from rest to momentum \vec{p} and the corresponding unitary operator acting on the particle states by $U[L(\vec{p})]$:

$$(18) \quad U[L(\vec{p})] = e^{-i\alpha \hat{p} \cdot \vec{K}} ,$$

where $\tanh \alpha = p/E$, $\sinh \alpha = p/w$, and $\cosh \alpha = E/w$.

A boost operator defines a Hermitian vector operator \vec{K} , and the components K_i are then the infinitesimal generators of “boosts”. The three components K_i together with J_i form the six infinitesimal generators of the homogeneous Lorentz group, and they satisfy definite commutation relations among them. We do not list the relations here, for they are not needed for our purposes.

From the relation $L(\vec{\beta})$ and the group property, one obtains

$$(19) \quad U[L(\vec{p})] = U[\overset{\circ}{R}(\phi, \theta, 0)]U[L_z(p)]U^{-1}[\overset{\circ}{R}(\phi, \theta, 0)] ,$$

where the rotation $\overset{\circ}{R}$ takes the z -axis into the direction of \vec{p} with spherical angles (θ, ϕ) :

$$(20) \quad \hat{p} = \overset{\circ}{R}(\phi, \theta, 0)\hat{z} .$$

We are now ready to define the “standard” or canonical state describing a single particle with spin j and momentum \vec{p} :

$$\begin{aligned}
 (21) \quad |\vec{p}, jm\rangle &= |\phi, \theta, p, jm\rangle = U[L(\vec{p})] |jm\rangle \\
 &= U[\overset{\circ}{R}(\phi, \theta, 0)] U[L_z(p)] U^{-1}[\overset{\circ}{R}(\phi, \theta, 0)] |jm\rangle,
 \end{aligned}$$

where $|jm\rangle$ is the particle state at rest as defined in the previous section. We emphasize that the z -component of spin m is measured in the rest frame of the particle and *not* in the frame where the particle has momentum \vec{p} .

The advantage of the **canonical state** is that the state transforms formally under rotation in the same way as the “rest-state” $|jm\rangle$:

$$\begin{aligned}
 (22) \quad U[R] |\vec{p}, jm\rangle &= U[R\overset{\circ}{R}] U[L_z(p)] U^{-1}[R\overset{\circ}{R}] U[R] |jm\rangle \\
 &= \sum_{m'} D_{m'm}^j(R) |R\vec{p}, jm'\rangle,
 \end{aligned}$$

It is clear from the rotational property of the canonical states that one may take over all the non-relativistic spin formalisms and apply them to situations involving relativistic particles with spin. One ought to remember, however, that the z -component of spin is defined only in the particle rest frame obtained from the frame where the particle has momentum \vec{p} via the pure Lorentz transformation $L^{-1}(\vec{p})$. See Fig. 1a.

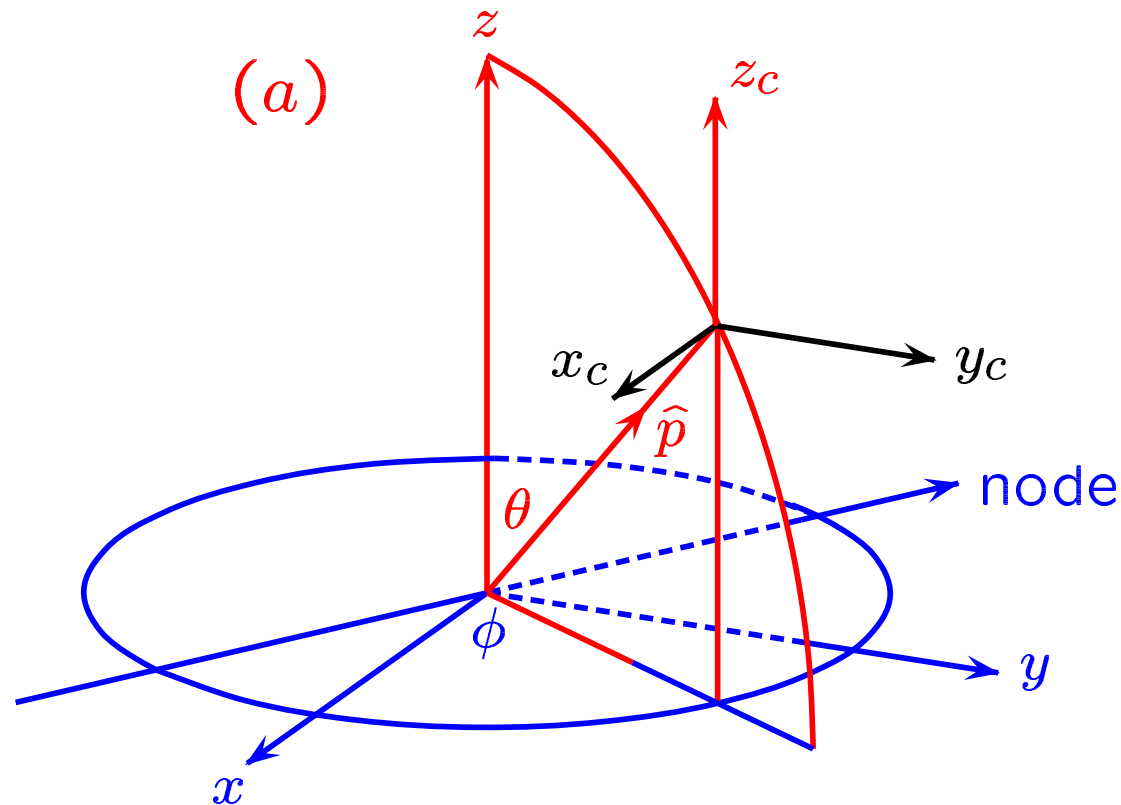


Fig. 1a: The orientation of the coordinate systems associated with a particle at rest in the canonical description $(\hat{x}_c, \hat{y}_c, \hat{z}_c)$.

Next, we shall define the helicity state describing a single particle with spin j and momentum \vec{p} [see Fig. 1b]:

$$\begin{aligned}
 (23) \quad |\vec{p}, j\lambda\rangle &= |\phi, \theta, p, j\lambda\rangle \\
 &= U[\overset{\circ}{R}(\phi, \theta, 0)] U[L_z(p)] |j\lambda\rangle = U[L(\vec{p})] U[\overset{\circ}{R}(\phi, \theta, 0)] |j\lambda\rangle
 \end{aligned}$$

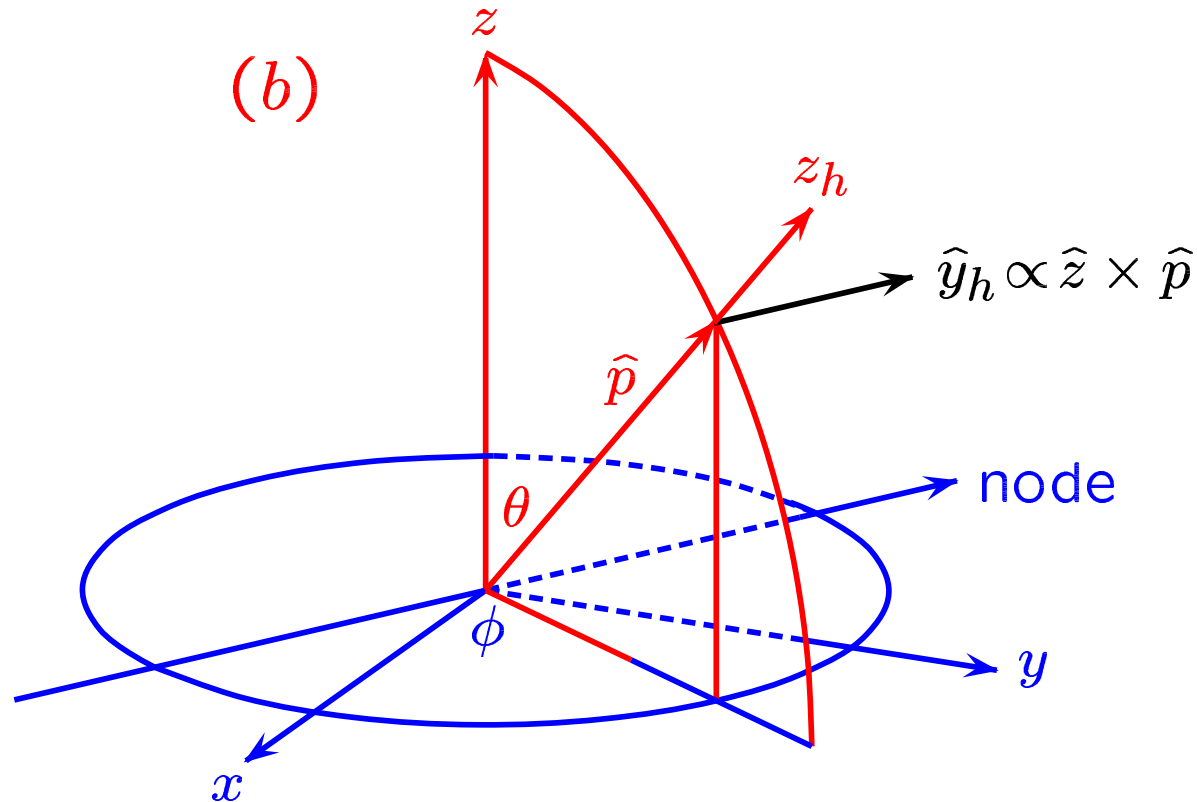


Fig. 1b: The orientation of the coordinate systems associated with a particle at rest in the helicity description ($\hat{x}_h = \hat{y}_h \times \hat{z}_h$, $\hat{y}_h \propto \hat{z} \times \hat{p}$, $\hat{z}_h = \hat{p}$).

Helicity states may be defined in two different ways. One may first rotate the rest state $|j\lambda\rangle$ by $\overset{\circ}{R}$, so that the quantization axis is along the \vec{p} direction and then boost the system along \vec{p} to obtain the helicity state $|\vec{p}, j\lambda\rangle$. Or, equivalently, one may first boost the rest state $|j\lambda\rangle$ along the z -axis and then rotate the system to obtain the state $|\vec{p}, j\lambda\rangle$. That these two different definitions of helicity state are equivalent is obvious from the relation for $U[L(\vec{p})]$ given previously.

One sees that, by definition, the helicity quantum number λ is the component of the spin along the momentum \vec{p} , and as such it is a rotationally invariant quantity, simply because the quantization axis itself rotates with the system under rotation. This fact may be seen easily from the definition of helicity states:

$$\begin{aligned}
 (24) \quad U[R] |\vec{p}, j\lambda\rangle &= U[R\overset{\circ}{R}] U[L_z] |j\lambda\rangle \\
 &= |R\vec{p}, j\lambda\rangle .
 \end{aligned}$$

In addition, the helicity λ remains invariant under pure Lorentz transformation that takes \vec{p} into \vec{p}' , which is parallel to \vec{p} . The invariance of λ under L' may be seen by

$$\begin{aligned}
 U[L'] |\vec{p}, j\lambda\rangle &= U[L'] U[L(\vec{p})] U[\overset{\circ}{R}] |j\lambda\rangle \\
 (25) \qquad \qquad &= U[L(\vec{p}')] U[\overset{\circ}{R}] |j\lambda\rangle \\
 &= |\vec{p}', j\lambda\rangle .
 \end{aligned}$$

There is a simple connection between the canonical and helicity descriptions. From the definitions of canonical and helicity states, one finds easily that

$$\begin{aligned}
 |\vec{p}, j\lambda\rangle &= U[\overset{\circ}{R}] U[L_z] U^{-1}[\overset{\circ}{R}] U[\overset{\circ}{R}] |j\lambda\rangle \\
 (26) \qquad &= \sum_m D_{m\lambda}^j(\overset{\circ}{R}) |\vec{p}, jm\rangle .
 \end{aligned}$$

We shall adopt here the following normalizations for the one-particle states:

$$(27) \quad \begin{aligned} \langle \vec{p}' j' m' | \vec{p} j m \rangle &= \tilde{\delta}(\vec{p}' - \vec{p}) \delta_{jj'} \delta_{mm'} \\ \langle \vec{p}' j' \lambda' | \vec{p} j \lambda \rangle &= \tilde{\delta}(\vec{p}' - \vec{p}) \delta_{jj'} \delta_{\lambda\lambda'} , \end{aligned}$$

where $\delta(\vec{p}' - \vec{p})$ is the Lorentz invariant δ -function given by

$$(28) \quad \tilde{\delta}(\vec{p}' - \vec{p}) = (2\pi)^3 (2E) \delta^{(3)}(\vec{p}' - \vec{p}) .$$

It can be shown that, with the invariant normalization given above, an arbitrary Lorentz transformation operator $U[\Lambda]$ acting on the states $|\vec{p}, jm\rangle$ or $|p, j\lambda\rangle$ is indeed a unitary operator, i.e. $U^\dagger U = I$. With the invariant volume element as defined by

$$(29) \quad \tilde{d}p = \frac{d^3\vec{p}}{(2\pi)^3 (2E)} ,$$

the completeness relations may be written as follows:

$$(30) \quad \sum_{jm} \int |\vec{p} jm\rangle \tilde{d}p \langle \vec{p} jm| = I, \quad \sum_{j\lambda} \int |\vec{p} j\lambda\rangle \tilde{d}p \langle \vec{p} j\lambda| = I ,$$

where I denotes the identity operator.

Parity and Time-Reversal Operations

Classically, the action of parity and time-reversal operations, denoted P and T , may be expressed as follows:

$$\begin{aligned} P: \quad & \vec{x} \rightarrow -\vec{x}, \quad \vec{p} \rightarrow -\vec{p}, \quad \vec{J} \rightarrow \vec{J} \\ T: \quad & \vec{x} \rightarrow \vec{x}, \quad \vec{p} \rightarrow -\vec{p}, \quad \vec{J} \rightarrow -\vec{J} \end{aligned} \tag{31}$$

where \vec{x} , \vec{p} , and \vec{J} stand for the coordinate, momentum, and angular momentum, respectively. It is clear that P and T commute with rotations, i.e.

$$(32) \quad [P, R] = 0, \quad [T, R] = 0 .$$

By definition, one sees also that the pure Lorentz transformations (in particular, boosts) act under P and T according to

$$(33) \quad PL(\vec{p}) = L(-\vec{p})P, \quad TL(\vec{p}) = L(-\vec{p})T .$$

Let us now define operators acting on the physical states, representing the parity and time-reversal operations:

$$(34) \quad \Pi = U[\mathbf{P}], \quad \mathbb{T} = \bar{U}[\mathbf{T}] ,$$

where Π is a unitary operator and \mathbb{T} is an anti-unitary (or anti-linear unitary) operator. \mathbf{T} is represented by an anti-unitary operator due to the fact that the time-reversal operation transforms an initial state into a final state and vice versa. Operators Π , \mathbb{T} , $U[R]$, and $U[L(\vec{p})]$ acting on the physical states should obey the same relations

$$(35) \quad \left[\Pi, U[R] \right] = 0 \quad \left[\mathbb{T}, U[R] \right] = 0$$

and

$$(36) \quad \begin{aligned} \Pi U[L(\vec{p})] &= U[L(-\vec{p})] \Pi \\ \mathbb{T} U[L(\vec{p})] &= U[L(-\vec{p})] \mathbb{T} . \end{aligned}$$

We are now ready to express the actions of Π and \mathbb{T} on the rest states $|jm\rangle$. It is clear that the quantum numbers j and m do not change under Π :

$$(37) \quad \Pi|jm\rangle = \eta|jm\rangle ,$$

where η is the intrinsic parity of the particle represented by $|jm\rangle$. Let us write the action of \mathbb{T} as follows:

$$\mathbb{T}|jm\rangle = \sum_k T_{km}|jk\rangle .$$

Remembering the anti-unitarity of \mathbb{T} ,

$$\sum_k D_{m'k}^j(R) T_{km} = \sum_k T_{m'k} D_{km}^{j*}(R) .$$

The above relation may be satisfied, if $T_{m'm}$ is given by

$$T_{m'm} = d_{m'm}^j(\pi) = (-)^{j-m} \delta_{m',-m} , \quad D_{m'm}^{j*}(\alpha, \beta, \gamma) = (-)^{m'-m} D_{-m'-m}^j(\alpha\beta\gamma)$$

so that the action of \mathbb{T} on the states $|jm\rangle$ may be expressed as

$$(38) \quad \mathbb{T}|jm\rangle = (-)^{j-m}|j-m\rangle .$$

One can show that the canonical state with momentum \vec{p} transforms under Π and \mathbb{T} as follows:

$$(39) \quad \begin{aligned} \Pi |\vec{p}, jm\rangle &= \eta |-\vec{p}, jm\rangle \\ \Pi |\phi, \theta, p, jm\rangle &= \eta |\pi + \phi, \pi - \theta, p, jm\rangle \end{aligned}$$

and

$$(40) \quad \begin{aligned} \mathbb{T} |\vec{p}, jm\rangle &= (-)^{j-m} |-\vec{p}, j-m\rangle \\ \mathbb{T} |\phi, \theta, p, jm\rangle &= (-)^{j-m} |\pi + \phi, \pi - \theta, p, j-m\rangle . \end{aligned}$$

Next, we wish to express the consequences of Π and \mathbb{T} operations on the helicity states $|p, j\lambda\rangle$. One finds

$$(41) \quad \Pi |\phi, \theta, p, j\lambda\rangle = \eta e^{-i\pi j} |\pi + \phi, \pi - \theta, p, j-\lambda\rangle$$

$$(42) \quad \mathbb{T} |\phi, \theta, p, j\lambda\rangle = e^{-i\pi\lambda} |\pi + \phi, \pi - \theta, p, j\lambda\rangle .$$

Now the helicity λ is an eigenvalue of $\vec{J} \cdot \hat{p}$. Note that $\vec{J} \cdot \hat{p} \rightarrow -\vec{J} \cdot \hat{p}$ under \mathbf{P} and $\vec{J} \cdot \hat{p} \rightarrow \vec{J} \cdot \hat{p}$ under \mathbf{T} . This explains why the helicity λ changes sign under Π , while it remains invariant under \mathbb{T} .

Two-Particle States

A system consisting of two particles with **arbitrary spins** may be constructed in two different ways; one using the **canonical** basis vectors $|\vec{p}, jm\rangle$, and the other using the **helicity** basis vectors $|\vec{p}, j\lambda\rangle$. We shall construct in this section both the canonical and helicity states for a two-particle system having definite spin and z -component, and then derive the **recoupling** coefficient which connects the two bases. Afterwards, we investigate the transformation properties of the two-particle states under **Π** and **\mathbb{T}** , as well as the consequences of the symmetrization required when the two particles are **identical**.

Construction of two-particle states:

Consider a system of two particles 1 and 2 with **intrinsic spins** s_1 and s_2 and **masses** w_1 and w_2 . In the two-particle rest frame, let \vec{p} be the momentum of the **particle 1**, with its direction given by the spherical angles (θ, ϕ) . we define the two-particle state in the **canonical** basis by

$$(43) \quad |\phi\theta m_1 m_2\rangle = a U[L(\vec{p})] |s_1 m_1\rangle U[L(-\vec{p})] |s_2 m_2\rangle ,$$

where $|s_i m_i\rangle$ is the rest-state of particle i and a is the **normalization constant** to be determined later. $L(\pm\vec{p})$ is the boost given by

$$(44) \quad L(\pm\vec{p}) = \overset{\circ}{R}(\phi, \theta, 0) L_{\pm z}(p) \overset{\circ}{R}^{-1}(\phi, \theta, 0) ,$$

where $\overset{\circ}{R}(\phi, \theta, 0)$ is again the rotation which carries the z -axis into the direction of \vec{p} and $L_{\pm z}(p)$ is the boost along the $\pm z$ -axis.

Owing to the rotational property of canonical one-particle states, one may define a state of **intrinsic total spin** s by

$$(45) \quad |\phi\theta sm_s\rangle = \sum_{m_1 m_2} (s_1 m_1 s_2 m_2 | sm_s) |\phi\theta m_1 m_2\rangle ,$$

where $(s_1 m_1 s_2 m_2 | sm_s)$ is the usual Clebsch-Gordan coefficient. Using the formula

$$(46) \quad D_{\mu_1 m_1}^{j_1} D_{\mu_2 m_2}^{j_2} = \sum_{j_3 \mu_3 m_3} (j_1 \mu_1 j_2 \mu_2 | j_3 \mu_3) (j_1 m_1 j_2 m_2 | j_3 m_3) D_{\mu_3 m_3}^{j_3}$$

and the **orthonormality** of the Clebsch-Gordan coefficients, one may easily show that, if R is a rotation which takes $\Omega = (\theta, \phi)$ into $R' = R\Omega$,

$$(47) \quad U[R] |\Omega sm_s\rangle = \sum_{m'_s} D_{m'_s m_s}^s(R) |R' sm'_s\rangle ,$$

so that the total spin s is a **rotational invariant**.

The state of a fixed orbital angular momentum is constructed in the usual way:

$$(48) \quad |\ell m s m_s\rangle = \int d\Omega Y_m^\ell(\Omega) |\Omega s m_s\rangle ,$$

where $d\Omega = d\phi d\cos\theta$. Let us investigate the rotational property of the formula above

$$(49) \quad U[R] |\ell m s m_s\rangle = \int d\Omega Y_m^\ell(\Omega) D_{m'_s m_s}^s(R) |R' s m'_s\rangle ,$$

where $R' = R'(\alpha', \beta', \gamma') = R\Omega$, $d\Omega = d\alpha' d\cos\beta'$, and, from

$$(50) \quad Y_m^\ell(\Omega) = \sqrt{\frac{2\ell+1}{4\pi}} D_{m0}^{\ell*}(\phi, \beta, 0)$$

we see that

$$(51) \quad \begin{aligned} Y_m^\ell(\Omega) &= \sqrt{\frac{2\ell+1}{4\pi}} D_{m0}^{\ell*}(R^{-1}R') \\ \text{(group property of } D\text{'s)} \rightarrow &= \sqrt{\frac{2\ell+1}{4\pi}} \sum_{m'} D_{mm'}^{\ell*}(R^{-1}) D_{m'0}^{\ell*}(R') \\ \text{(unitarity of } D\text{'s)} \rightarrow &= \sum_{m'} D_{m'm}^\ell(R) Y_{m'}^\ell(\beta', \alpha') , \end{aligned}$$

one obtains the result

$$(52) \quad U[R] |\ell m s m_s\rangle = \sum_{m' m'_s} D_{m' m}^{\ell}(R) D_{m'_s m_s}^s(R) |\ell m' s m'_s\rangle .$$

This shows that the states $|\ell m s m_s\rangle$ transform under rotation as a **product of two “rest states”** $|\ell m\rangle$ and $|s m_s\rangle$.

Now, it is easy to construct a state of **total angular momentum** J :

$$\begin{aligned}
 (53) \quad |JM\ell s\rangle &= \sum_{mm_s} (\ell m \ s m_s | JM) |\ell m s m_s\rangle \\
 &= \sum_{\substack{mm_s \\ m_1 m_2}} (\ell m \ s m_s | JM) (s_1 m_1 \ s_2 m_2 | s m_s) \times \\
 &\quad \times \int d\Omega Y_m^\ell(\Omega) |\Omega m_1 m_2\rangle .
 \end{aligned}$$

One sees immediately

$$(54) \quad U[R] |JM\ell s\rangle = \sum_{M'} D_{M'M}^J(R) |JM'\ell s\rangle .$$

Note that, as expected, ℓ and s are **rotational invariants**: This is the equivalent of the non-relativistic L - S coupling.

Next, we turn to the problem of constructing two-particle states from the helicity basis vectors $|\vec{p}, j\lambda\rangle$. we write

$$\begin{aligned}
 (55) \quad |\phi\theta\lambda_1\lambda_2\rangle &= a U[\overset{\circ}{R}] \left\{ U[L_z(p)] |s_1\lambda_1\rangle U[L_{-z}(p)] |s_2 -\lambda_2\rangle \right\} \\
 &\equiv U[\overset{\circ}{R}(\phi, \theta, 0)] |00\lambda_1\lambda_2\rangle ,
 \end{aligned}$$

where $|s_i\lambda_i\rangle$ is the rest state of particle i and a the **normalization constant** introduced previously. We have constructed the helicity state for the particle 2 in such a way that **its helicity quantum number is $+\lambda_2$** .

States of a definite **total angular momentum J** may be constructed as follows:

$$(56) \quad |JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi} \int dR D_{M\mu}^{J*}(R) U[R] |00\lambda_1\lambda_2\rangle ,$$

where N_J is a normalization constant to be determined later. Let us apply an arbitrary rotation R' on this state

$$U[R'] |JM\lambda_1\lambda_2\rangle = \frac{N_J}{2\pi} \int dR D_{M\mu}^{J*}(R) U[R'] |00\lambda_1\lambda_2\rangle ,$$

where $R'' = R'R$. But, by using the multiplication law and the unitarity of the D -functions,

$$(57) \quad \begin{aligned} D_{M\mu}^{J*}(R) &= D_{M\mu}^{J*}(R'^{-1}R'') \\ &= \sum_{M'} D_{MM'}^{J*}(R'^{-1}) D_{M'\mu}^{J*}(R'') \\ \text{Unitarity} \rightarrow &= \sum_{M'} D_{M'M}^J(R') D_{M'\mu}^{J*}(R'') . \end{aligned}$$

Using this relation, as well as the fact that $dR = dR''$, one obtains the result

$$(58) \quad U[R'] |JM\lambda_1\lambda_2\rangle = \sum_{M'} D_{M'M}^J(R') |JM'\lambda_1\lambda_2\rangle ,$$

so that the states are indeed states of a definite angular momentum J . Note that, as expected, λ_1 and λ_2 are rotational invariants.

Now, let us specify the rotation R by writing $R = R(\phi, \theta, \gamma)$. Then,

$$(59) \quad \begin{aligned} U[R(\phi, \theta, \gamma)] |00\lambda_1\lambda_2\rangle &= U[R(\phi, \theta, 0)]U[R(0, 0, \gamma)] |00\lambda_1\lambda_2\rangle \\ &= e^{-i(\lambda_1 - \lambda_2)\gamma} U[R(\phi, \theta, 0)] |00\lambda_1\lambda_2\rangle . \end{aligned}$$

The last relation follows because of the commutation relation: $[R(0, 0, \gamma), L_{\pm z}(p)] = 0$. Integrating over $d\gamma$, one obtains

$$(60) \quad |JM\lambda_1\lambda_2\rangle = N_J \int d\Omega D_{M\lambda}^{J*}(\phi, \theta, 0) |\phi\theta\lambda_1\lambda_2\rangle ,$$

where $\lambda = \lambda_1 - \lambda_2$.

Normalization:

For simplicity of notation, **we shall deal with spinless particles**. Two-particle states are normalized

$$(61) \quad \langle \vec{p}'_1 \vec{p}'_2 | \vec{p}_1 \vec{p}_2 \rangle = \tilde{\delta}(\vec{p}'_1 - \vec{p}_1) \tilde{\delta}(\vec{p}'_2 - \vec{p}_2) ,$$

where

$$(62) \quad \tilde{\delta}(\vec{p}'_i - \vec{p}_i) = (2\pi)^3 (2E_i) \delta^{(3)}(\vec{p}'_i - \vec{p}_i) , \quad i = 1 \text{ or } 2 .$$

A system consisting of two momenta \vec{p}_1 and \vec{p}_2 may be described, in general, by one four-momentum p representing the sum of the four-momenta of particles 1 and 2 and Ω describing the orientation of the relative momentum in the (1,2) rest frame, i.e.

$$(63) \quad |p, \Omega\rangle = a |\vec{p}_1 \vec{p}_2\rangle , \quad \text{where } p = p_1 + p_2 \quad \text{and} \quad p^2 = w^2 .$$

We adopt the normalization for this state as follows:

$$(64) \quad \langle p', \Omega' | p, \Omega \rangle = (2\pi)^4 \delta^{(4)}(p' - p) \delta^{(2)}(\Omega' - \Omega) .$$

Multiply the formula above by the invariant volume element $\tilde{d}p_1 \tilde{d}p_2$, where

$$(65) \quad \tilde{d}p = \frac{d^3\vec{p}}{(2\pi)^3 (2E)} ,$$

and integrate over these variables.

Using the notation for which $d\phi_n$ is the n -body phase space so that

$$(66) \quad d\phi_2(1, 2) = (2\pi)^4 \delta^{(4)}(p' - p) \tilde{d}p_1 \tilde{d}p_2$$

we need to evaluate an integration, **in the two-particle rest frame (RF)**,

$$(67) \quad a^2 = \int \delta^{(2)}(\Omega' - \Omega) d\phi_2(1, 2)$$

$$\left(\int d^3\vec{p}_2; \vec{p}_1 \rightarrow \vec{p} \right) \rightarrow = \frac{1}{(2\pi)^2} \int \delta(E_1 + E_2 - w) \delta^{(2)}(\Omega' - \Omega) \frac{p^2 dp d\Omega}{4E_1 E_2}$$

$$\left(\int d\Omega; E_i = \sqrt{p^2 + w_i^2} \right) \rightarrow = \frac{1}{(4\pi)^2} \frac{p}{w}$$

where p is **the relative momentum in the two-particle RF** and $w = E_1 + E_2$ is the effective mass of the two-particle system. We see immediately that

$$(68) \quad a = \frac{1}{4\pi} \sqrt{\frac{p}{w}},$$

We are now ready to specify the normalizations for **the two-particle states with arbitrary spin.**

$$(69) \quad \langle \Omega' m'_1 m'_2 | \Omega m_1 m_2 \rangle = \delta^{(2)}(\Omega' - \Omega) \delta_{m_1 m'_1} \delta_{m_2 m'_2}$$

and

$$(70) \quad \langle \Omega' \lambda'_1 \lambda'_2 | \Omega \lambda_1 \lambda_2 \rangle = \delta^{(2)}(\Omega' - \Omega) \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} .$$

The states $|JM\ell s\rangle$ obey the following normalizations:

$$(71) \quad \langle J' M' \ell' s' | JM\ell s \rangle = \delta_{JJ'} \delta_{MM'} \delta_{\ell\ell'} \delta_{ss'} .$$

Since the D -functions are normalized according to

$$(72) \quad \int dR D_{\mu_1 m_1}^{j_1 *}(R) D_{\mu_2 m_2}^{j_2}(R) = \frac{8\pi^2}{2j_1 + 1} \delta_{j_1 j_2} \delta_{\mu_1 \mu_2} \delta_{m_1 m_2} ,$$

where $R = R(\alpha, \beta, \gamma)$ and $dR = d\alpha d \cos \beta d\gamma$, the state $|JM\lambda_1\lambda_2\rangle$ is seen to be normalized according to

$$(73) \quad \langle J' M' \lambda'_1 \lambda'_2 | JM \lambda_1 \lambda_2 \rangle = \delta_{JJ'} \delta_{MM'} \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} ,$$

if the constant N_J is set equal to

$$(74) \quad N_J = \sqrt{\frac{2J + 1}{4\pi}} .$$

The completeness relations may now be written

$$(75) \quad \sum_{\substack{JM \\ \ell s}} |JM\ell s\rangle \langle JM\ell s| = I$$

and

$$(76) \quad \sum_{\substack{JM \\ \lambda_1 \lambda_2}} |JM\lambda_1 \lambda_2\rangle \langle JM\lambda_1 \lambda_2| = I .$$

Finally, we note the relation

$$(77) \quad \langle \Omega \lambda'_1 \lambda'_2 | JM \lambda_1 \lambda_2 \rangle = N_J D_{M\lambda}^{J*}(\phi, \theta, 0) \delta_{\lambda_1 \lambda'_1} \delta_{\lambda_2 \lambda'_2} .$$

Connection between canonical and helicity states:

We start from

$$\begin{aligned}
 |\phi\theta\lambda_1\lambda_2\rangle &= a U[\mathring{R}] \left\{ U[L_z(p)] |s_1\lambda_1\rangle U[L_{-z}(p)] |s_2-\lambda_2\rangle \right\} \\
 (78) \qquad &= a U[L(\vec{p})] U[\mathring{R}] |s_1\lambda_1\rangle U[L(-\vec{p})] U[\mathring{R}] |s_2-\lambda_2\rangle \\
 &= \sum_{m_1 m_2} D_{m_1\lambda_1}^{s_1}(\phi, \theta, 0) D_{m_2-\lambda_2}^{s_2}(\phi, \theta, 0) |\phi\theta m_1 m_2\rangle,
 \end{aligned}$$

we see that

$$(79) \qquad |JM\lambda_1\lambda_2\rangle = N_J \sum_{m_1 m_2} \int d\Omega D_{M\lambda}^{J*}(\phi, \theta, 0) D_{m_1\lambda_1}^{s_1}(\phi, \theta, 0) D_{m_2-\lambda_2}^{s_2}(\phi, \theta, 0) |\phi\theta m_1 m_2\rangle.$$

The product of three D -functions appearing may be reduced as follows:

$$(80) \qquad D_{m_1\lambda_1}^{s_1} D_{m_2-\lambda_2}^{s_2} = \sum_{sm_s} (s_1 m_1 s_2 m_2 | sm_s) (s_1 \lambda_1 s_2 - \lambda_2 | s \lambda) D_{m_s \lambda}^s$$

and

$$(81) \quad D_{M\lambda}^{J*} D_{m_s\lambda}^s = \sum_{\ell m} \sqrt{\frac{4\pi}{2\ell+1}} \left(\frac{2\ell+1}{2J+1} \right) (\ell m \ s m_s | JM) (\ell 0 \ s\lambda | J\lambda) Y_m^\ell .$$

we obtain finally

$$(82) \quad |JM\lambda_1\lambda_2\rangle = \sum_{\ell s} \left(\frac{2\ell+1}{2J+1} \right)^{\frac{1}{2}} (\ell 0 \ s\lambda | J\lambda) (s_1\lambda_1 \ s_2 -\lambda_2 | s\lambda) |JM\ell s\rangle ,$$

so that the recoupling coefficient between canonical and helicity states is given by

$$(83) \quad \langle J'M'\ell s | JM\lambda_1\lambda_2 \rangle = \left(\frac{2\ell+1}{2J+1} \right)^{\frac{1}{2}} (\ell 0 \ s\lambda | J\lambda) (s_1\lambda_1 \ s_2 -\lambda_2 | s\lambda) \delta_{JJ'} \delta_{MM'} .$$

This relation may be inverted to give

$$(84) \quad \begin{aligned} |JM\ell s\rangle &= \sum_{\lambda_1\lambda_2} |JM\lambda_1\lambda_2\rangle \langle JM\lambda_1\lambda_2 | JM\ell s\rangle \\ &= \sum_{\lambda_1\lambda_2} \left(\frac{2\ell+1}{2J+1} \right)^{\frac{1}{2}} (\ell 0 \ s\lambda | J\lambda) (s_1\lambda_1 \ s_2 -\lambda_2 | s\lambda) |JM\lambda_1\lambda_2\rangle . \end{aligned}$$

Symmetry relations:

The canonical states $|JM\ell s\rangle$ transform in a particularly simple manner under symmetry operations (e.g. parity and time-reversal), and the derivation is also much simpler than for helicity states. For this reason, we shall first investigate the consequences of symmetry operations on the canonical states, and then obtain the corresponding relations for the states $|JM\lambda_1\lambda_2\rangle$.

We shall first start with the parity operation. We find

$$(85) \quad \Pi|\phi\theta m_1 m_2\rangle = \eta_1\eta_2|\pi + \phi, \pi - \theta, m_1 m_2\rangle ,$$

where $\eta_1(\eta_2)$ is the intrinsic parity of particle 1(2). We then obtain immediately

$$(86) \quad Y_m^\ell(\pi - \theta, \pi + \phi) = (-)^\ell Y_m^\ell(\theta, \phi), \quad \Pi|JM\ell s\rangle = \eta_1\eta_2(-)^\ell|JM\ell s\rangle ,$$

so that the “ ℓ - s coupled” states are in an eigenstate of Π with the eigenvalue $\eta_1\eta_2(-)^\ell$, a well known result. Using the recoupling formula and the symmetry relations of Clebsch-Gordan coefficients, one finds for the helicity states

$$(87) \quad \Pi|JM\lambda_1\lambda_2\rangle = \eta_1\eta_2(-)^{J-s_1-s_2}|JM-\lambda_1-\lambda_2\rangle .$$

Again, the helicities reverse sign, as was the case for the single-particle states.

Consequences of the time-reversal operation may be explored in a similar fashion. one finds

$$(88) \quad \mathbb{T} |\phi\theta m_1 m_2\rangle = (-)^{s_1 - m_1} (-)^{s_2 - m_2} |\pi + \phi, \pi - \theta, -m_1 - m_2\rangle .$$

and so

$$(89) \quad \mathbb{T} |JM\ell s\rangle = (-)^{J-M} |J - M\ell s\rangle$$

and

$$(90) \quad \mathbb{T} |JM\lambda_1\lambda_2\rangle = (-)^{J-M} |J - M\lambda_1\lambda_2\rangle .$$

Now, we investigate the effects of symmetrization required when the particles 1 and 2 are identical— **No field theory here**—

$$(91) \quad |JM\ell s\rangle_s = a_s [1 + (-)^{2s_1} \mathbb{P}_{12}] |JM\ell s\rangle, \quad \mathbb{P}_{12} |JM\ell s\rangle = (-)^{\ell+s-2s_1} |JM\ell s\rangle$$

where \mathbb{P}_{12} is **the particle-exchange operator** and a_s is the normalization constant. Again, using the defining equation, one obtains

$$(92) \quad |JM\ell s\rangle_s = a_s [1 + (-)^{\ell+s}] |JM\ell s\rangle,$$

so that $\ell + s = \text{even}$ for a system of identical particles in an eigenstate of orbital angular momentum ℓ and total spin s and $a_s = 1/2$. Now, the symmetrized helicity state may be written

$$(93) \quad |JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2) [1 + (-)^{2s_1} \mathbb{P}_{12}] |JM\lambda_1\lambda_2\rangle$$

where $b_s(\lambda_1\lambda_2)$ is another normalization constant. Using $|JM\ell s\rangle_s$, one finds

$$(94) \quad |JM\lambda_1\lambda_2\rangle_s = b_s(\lambda_1\lambda_2) \left\{ |JM\lambda_1\lambda_2\rangle + (-)^J |JM\lambda_2\lambda_1\rangle \right\},$$

where $b_s(\lambda_1\lambda_2) = 1/\sqrt{2}$ for $\lambda_1 \neq \lambda_2$ and $b_s(\lambda_1\lambda_2) = 1/2$ for $\lambda_1 = \lambda_2$. Note that, for a system of identical particles, the symmetrized states in both **canonical and helicity bases** have the **same forms**, regardless of whether the particles involved are **fermions or bosons**.

