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# Hadronic Contributions to the Photon Vacuum Polarization and their Role in Precision Physics

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I review recent evaluations of the hadronic contribution to the shift in the fine structure constant and to the anomalous magnetic moment of the muon. Substantial progress in a precise determination of these important observables is a consequence of substantially improved total cross section measurement by the BES II collaboration and an improved theoretical understanding. Prospects of further possible progress is discussed.

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(enhanced by 3 figures and an addendum)

# 1 Introduction

Precision physics requires appropriate inclusion of higher order effects and the knowledge of very precise input parameters of the electroweak standard model SM. One of the basic input parameters is the fine structure constant which depends logarithmically on the energy scale. Vacuum polarization effects lead to a partial screening of the charge in the low energy limit (Thomson limit) while at higher energies the strength of the electromagnetic interaction grows. We discuss the current status of the hadronic contributions to some electroweak precision observables like the leading hadronic contribution to the muon anomalous magnetic moment $a_\mu \equiv (g_\mu - 2)/2$ [1] and the effective fine structure constant at the $Z$–resonance [2].

Renormalization of the electric charge $e$ by a shift $\delta e$ at different scales leads to a shift of the fine structure constant by

$$\Delta\alpha = 2\left(\frac{\delta e}{e}(0) - \frac{\delta e}{e}(M_Z)\right) = \Pi'_\gamma(0) - \Pi'_\gamma(M_Z^2) \tag{1}$$

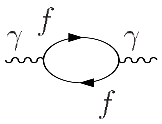
where $\Pi'_\gamma(s)$ is the photon vacuum polarization function defined via the time-ordered product of two electromagnetic currents $j^\mu_{em}(x)$:

$$i\int d^4x\, e^{iq\cdot x}\langle 0|\mathrm{T}j^\mu_{em}(x)j^\nu_{em}(0)|0\rangle = -(q^2g^{\mu\nu} - q^\mu q^\nu)\Pi'_\gamma(q^2)\ . \tag{2}$$

The shift $\Delta\alpha$ is large due to the large change in scale going from zero momentum to the Z-mass scale $\mu = M_Z$ and due to the many species of fermions contributing. Zero momentum more precisely means the light fermion mass thresholds.

In perturbation theory the leading light fermion ($m_f \ll M_Z$) contribution is given by

$$\Delta\alpha = \sum_f$$




$$\begin{aligned} &= \frac{\alpha}{3\pi}\sum_f Q_f^2 N_{cf}\left(\ln\frac{M_Z^2}{m_f^2} - \frac{5}{3}\right) \\ &= \Delta\alpha_{\mathrm{leptons}} + \Delta\alpha^{(5)}_{\mathrm{quarks}}\ . \end{aligned} \tag{3}$$

A serious problem is the low energy contributions of the five light quarks u,d,s,c and b which cannot be reliably calculated using perturbative quantum chromodynamics (p-QCD). The evaluation of the hadronic contribution $\Delta\alpha^{(5)}_{\mathrm{quarks}} \to \Delta\alpha^{(5)}_{\mathrm{hadrons}}$ is the main concern of this mini review. Before I am going into this, let me make a few remarks about the consequences of the related problems for precision physics.

A major drawback of the partially non-perturbative relationship between $\alpha(0)$ and $\alpha(M_Z)$ is that one has to rely on experimental data exhibiting systematic and statistical errors which implies a non-negligible uncertainty in our knowledge of the effective fine structure constant. In precision predictions of gauge boson properties this has become a limiting factor. Since $\alpha$ , $G_\mu, M_Z$ are the most precisely measured parameters, they are used as input parameters for accurate predictions of observables like the effective weak mixing parameter $\sin^2\Theta_f$, the vector $v_f$ and axial-vector $a_f$ neutral current couplings, the $W$ mass $M_W$ the widths $\Gamma_Z$ and $\Gamma_W$ of the $Z$ and the $W$, respectively, etc. However, for physics at higher energies we have to use the effective couplings at the appropriate scale, for physics at the $Z$–resonance, for example, $\alpha(M_Z)$ is more adequate to use than $\alpha(0)$. Of course this just means that part of the higher order corrections may be absorbed into an effective parameter. If we compare the precision of the basic parameters

$$\begin{array}{cccccccccc} \frac{\delta\alpha}{\alpha} & \sim & 3.6 & \times & 10^{-9} & \frac{\delta\alpha(M_Z)}{\alpha(M_Z)} & \sim & 1.6 \div 6.8 & \times & 10^{-4} \\ \frac{\delta G_\mu}{G_\mu} & \sim & 8.6 & \times & 10^{-6} & \frac{\delta M_Z}{M_Z} & \sim & 2.4 & \times & 10^{-5} \end{array} \tag{4}$$

we observe that the uncertainty in $\alpha(M_Z)$ is roughly an order of magnitude worse than the next best, which is the $Z$–mass. Let me remind the reader that $\Delta\alpha$ enters in electroweak precision physics typically when calculating versions of the weak mixing parameter $\sin^2\Theta_i$ from $\alpha$, $G_\mu$ and $M_Z$ via

$$\sin^2\Theta_i \, \cos^2\Theta_i \; = \frac{\pi\,\alpha}{\sqrt{2}\,G_\mu\,M_Z^2}\frac{1}{1-\Delta r_i} \; . \tag{5}$$

where

$$\Delta r_i \;\; = \;\; \Delta r_i(\alpha,\, G_\mu,\; M_Z, m_H,\; m_{f\neq t},\; m_t) \tag{6}$$

includes the higher order corrections which can be calculated in the SM or in alternative models. It has been calculated for the first time by Alberto Sirlin in 1980 [3]. In the SM today the Higgs mass $m_H$ is the only relevant unknown parameter and by confronting the calculated with the experimentally determined value of $\sin^2\Theta_i$ one obtains important indirect constraints on the Higgs mass. $\Delta r_i$ depends on the definition of $\sin^2\Theta_i$. The various definitions coincide at tree level and hence only differ by quantum effects. From the weak gauge boson masses, the electroweak gauge couplings and the neutral current couplings of the charged fermions we obtain

$$\sin^2\Theta_W \;\; = \;\; 1-\frac{M_W^2}{M_Z^2} \tag{7}$$

$$\sin^2\Theta_g \;\; = \;\; e^2/g^2 = \frac{\pi\,\alpha}{\sqrt{2}\,G_\mu\,M_W^2} \tag{8}$$

$$\sin^2\Theta_f \;\; = \;\; \frac{1}{4|Q_f|}\left(1-\frac{v_f}{a_f}\right)\;,\;\; f\neq\nu\;, \tag{9}$$

for the most important cases and the general form of $\Delta r_i$ reads

$$\Delta r_i \;\; = \;\; \Delta\alpha - f_i(\sin^2\Theta_i)\,\Delta\rho + \Delta r_{i\,\mathrm{reminder}} \tag{10}$$

with a universal term $\Delta\alpha$ which affects the predictions of $M_W$, $A_{LR}$, $A_{FB}^f$, $\Gamma_f$, etc. The uncertainty $\delta\Delta\alpha$ implies uncertainties $\delta M_W$, $\delta \sin^2\Theta_i$ given by

$$\frac{\delta M_W}{M_W} \sim \frac{1}{2}\frac{\sin^2\Theta_W}{\cos^2\Theta_W-\sin^2\Theta_W}\,\delta\Delta\alpha \sim 0.23\;\delta\Delta\alpha \tag{11}$$

$$\frac{\delta\sin^2\Theta_f}{\sin^2\Theta_f} \sim \frac{\cos^2\Theta_f}{\cos^2\Theta_f-\sin^2\Theta_f}\,\delta\Delta\alpha \sim 1.54\;\delta\Delta\alpha \tag{12}$$

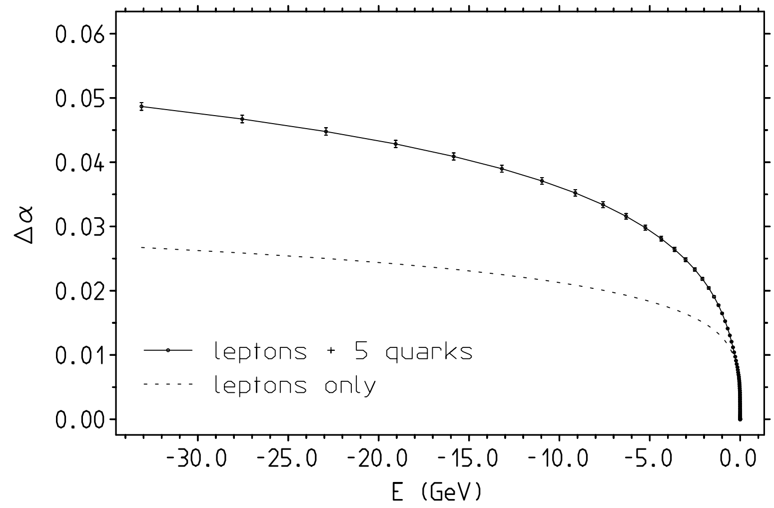



Figure 1: The running of $\alpha$. The "negative" $E$ axis is chosen to indicate space-like momentum transfer. The vertical bars at selected points indicate the uncertainty.

which obscure in particular the indirect bounds on the Higgs mass obtained from electroweak precision measurements. A summary of the present status and future expectations will be presented below. Once the Higgs boson will have been discovered and its mass is known, precision measurements of the $\Delta r_i$, which would be possible with the GigaZ option of TESLA [4], would provide excellent possibilities to establish new physics contributions beyond the SM.

## 2 The hadronic contributions to $\alpha(s)$

The effective QED coupling constant at scale $\sqrt{s}$ is given by the renormalization group resummed running fine structure constant

$$\alpha(s) = \frac{\alpha}{1-\Delta\alpha(s)} \tag{13}$$

with

$$\Delta\alpha(s) = -4\pi\alpha\mathrm{Re}\left[\Pi'_\gamma(s)-\Pi'_\gamma(0)\right]\ . \tag{14}$$

Fig. 1 illustrates the running of the effective charges at lower energies in the space-like region. Typical values are $\Delta\alpha(5\mathrm{GeV}) \sim 3\%$ and $\Delta\alpha(M_Z) \sim 6\%$, where about $\sim 50\%$ of the contribution comes from leptons and about $\sim 50\%$ from hadrons.

The leptonic contributions are calculable in perturbation theory where at leading order the free lepton loops yield

$$\begin{aligned}
&\Delta\alpha_{\mathrm{leptons}}(s) = \\
&= \sum_{\ell=e,\mu,\tau} \tfrac{\alpha}{3\pi}\left[-\tfrac{8}{3}+\beta_\ell^2-\tfrac{1}{2}\beta_\ell(3-\beta_\ell^2)\ln\left(\tfrac{1-\beta_\ell}{1+\beta_\ell}\right)\right] \\
&= \sum_{\ell=e,\mu,\tau} \tfrac{\alpha}{3\pi}\left[\ln\left(s/m_\ell^2\right)-\tfrac{5}{3}+O\left(m_\ell^2/s\right)\right] \text{ for } |s| \gg m_\ell^2 \\
&\simeq 0.03142 \text{ for } s = M_Z^2
\end{aligned} \tag{15}$$

where $\beta_\ell = \sqrt{1-4m_\ell^2/s}$. This leading contribution is affected by small electromagnetic corrections only in the next to leading order. The leptonic contribution is actually known to three loops [5,6] at which it takes the value

$$\Delta\alpha_{\mathrm{leptons}}(M_Z^2) \simeq 314.98 \times 10^{-4}. \tag{16}$$

In contrast, the corresponding free quark loop contribution gets substantially modified by low energy strong interaction effects, which cannot be obtained by p-QCD. Fortunately, one can evaluate this hadronic term $\Delta\alpha^{(5)}_{\mathrm{hadrons}}$ from hadronic $e^+e^-$- annihilation data by using a dispersion relation. The relevant vacuum polarization amplitude satisfies the convergent dispersion relation

$$Re\Pi'_\gamma(s) - \Pi'_\gamma(0) = \frac{s}{\pi} Re \int_{s_0}^{\infty} ds' \frac{Im\Pi'_\gamma(s')}{s'(s'-s-i\varepsilon)}$$

and using the optical theorem (unitarity) one has

$$Im\Pi'_\gamma(s) = \frac{s}{e^2}\sigma_{tot}(e^+e^- \to \gamma^* \to \mathrm{hadrons})(s)\ .$$

In terms of the cross-section ratio

$$R(s) = \frac{\sigma_{tot}(e^+e^- \to \gamma^* \to \mathrm{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)}\ , \tag{17}$$

where $\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3s}$ at tree level, we finally obtain

$$\Delta\alpha^{(5)}_{\mathrm{hadrons}}(M_Z^2) = -\frac{\alpha M_Z^2}{3\pi} Re \int_{4m_\pi^2}^{\infty} ds \frac{R(s)}{s(s-M_Z^2-i\varepsilon)}\ . \tag{18}$$

Using the experimental data for $R(s)$ up to $\sqrt{s} = E_{cut} = 5.5$ GeV and for the $\Upsilon$ resonances region between 9.6 and 11 GeV and perturbative QCD from 5.5 to 9.6

GeV and for the high energy tail [7,8,9] above 11 GeV we get as an update of [10] including the recent new data from CMD [11] and BES [12]

$$\Delta\alpha^{(5)}_{\rm hadrons}(M_Z^2) \quad = \quad 0.027896 \pm 0.000395$$

with $M_Z = 91.19$ GeV [13]. The CMD-2 experiment at Novosibirsk has continued and substantially improved the $\sigma(e^+e^- \to \text{hadrons})$ measurements below 1.4 GeV [11] and the BES II experiment at Beijing has published a new measurement, which in the region from 2 to 5 GeV improves the evaluation from 15% to 20% systematic error to about 6.6% [12]. As a consequence we observe a dramatic reduction of the error with respect to our 1995 evaluation $0.0280 \pm 0.0007$ [10] mainly due to the new BES data. For an evaluation which yields a quite different answer see [14] and Tab. 1 below.

## 3 Theoretical progress

Because of the large uncertainties in the data many authors advocated to extend the use of perturbative QCD in place of data [15,16,17,18,19]. The assumption that p-QCD may be reliable to calculate $R(s)$ down to energies as low as 1.8 GeV seems to be supported by

- the apparent applicability of p-QCD to $\tau$ physics. In fact the running of $\alpha_s(M_\tau) \to \alpha_s(M_Z)$ from the $\tau$ mass up to LEP energies agrees well with the LEP value. The estimated uncertainty may be debated, however.
- the smallness of non–perturbative (NP) effects [18] (see also: [20]) if parametrized as prescribed by the operator product expansion (OPE) of the electromagnetic current correlator [21].

Progress in p-QCD comes mainly from [22]. In addition an exact two–loop calculation of the renormalization group (RG) in the gauge invariant background field MOM scheme is now available [23] which allows us to treat "threshold effects" closer to physics than in the $\overline{\rm MS}$ scheme. Except from Ref. [16] which is based on [22] most other "improved" calculations utilize older results, mainly, the well known massless result [7] plus some leading mass corrections. For a recent critical review of the newer estimates of vacuum polarization effects see [24] and Tab.1 below.

In Ref. [25] a different approach of p-QCD improvement was proposed, which relies on the fact that the vacuum polarization amplitude $\Pi(q^2)$ is an analytic function in $q^2$ with a cut in the $s$–channel $q^2 = s \geq 0$ at $s \geq 4m_\pi^2$ and a smooth behavior in the $t$–channel (space-like or Euclidean region). Thus, instead of trying to calculate the complicated function $R(s)$, which obviously exhibits non-perturbative features like resonances, one considers the simpler Adler function in the Euclidean region. In [25] the Adler function was investigated and p-QCD was found to work very well above

2.5 GeV, provided the exact three–loop mass dependence was used (in conjunction with the background field MOM scheme). The Adler function may be defined as a derivative

$$D(-s) = -(12\pi^2)\, s\, \frac{d\Pi'_\gamma(s)}{ds} = \frac{3\pi}{\alpha} s \frac{d}{ds} \Delta\alpha_{\mathrm{had}}(s) \tag{19}$$

of (17) which is the hadronic contribution to the shift of the fine structure constant. It is represented by

$$D(Q^2) = Q^2 \left( \int_{4m_\pi^2}^{E^2_{\mathrm{cut}}} \frac{R^{\mathrm{data}}(s)}{(s+Q^2)^2} ds \; + \; \int_{E^2_{\mathrm{cut}}}^{\infty} \frac{R^{\mathrm{pQCD}}(s)}{(s+Q^2)^2} ds \right) \tag{20}$$

in terms of the experimental $e^+e^-$–data. The standard evaluation ([10]) of (20) then yields the non–perturbative "experimental" Adler function, as displayed in Fig. 2.

For the p-QCD evaluation it is mandatory to utilize the calculations with massive quarks which are available up to three–loops [22]. The four-loop corrections are known in the approximation of massless quarks [7]. The outcome of this analysis is pretty surprising and is shown in Fig. 2. For a discussion we refer to the original paper [25].

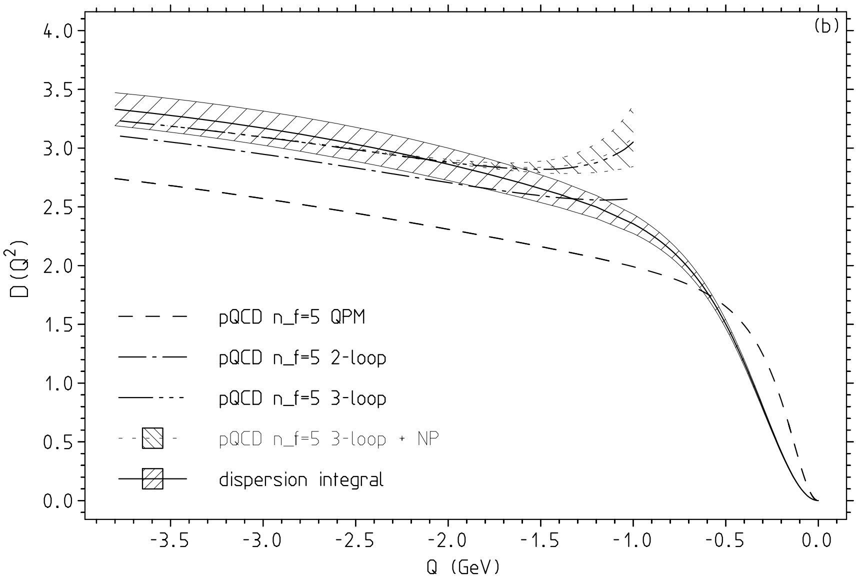



Figure 2: The Adler function: theory vs. experiment.

According to (19), we may compute the hadronic vacuum polarization contribution to the shift in the fine structure constant by integrating the Adler function. In the region where p-QCD works fine we integrate the p-QCD prediction, in place of the data. We thus calculate in the Euclidean region

$$\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) = \left[\Delta\alpha^{(5)}_{\mathrm{had}}(-M_Z^2) - \Delta\alpha^{(5)}_{\mathrm{had}}(-s_0)\right]^{\mathrm{p-QCD}} + \Delta\alpha^{(5)}_{\mathrm{had}}(-s_0)^{\mathrm{data}} \quad . \tag{21}$$

A save choice is $s_0 = (2.5\,\mathrm{GeV})^2$ where we obtain $\Delta\alpha^{(5)}_{\rm had}(-s_0)^{\rm data} = 0.007489 \pm 0.000146$ from the evaluation of the dispersion integral (17). With the results presented above we find [13]

$$\Delta\alpha^{(5)}_{\rm had}(-M_Z^2) = 0.027685 \pm 0.000146^* \tag{22}$$

for the Euclidean effective fine structure constant. Adding $\Delta\alpha^{(5)}_{\rm had}(M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(-M_Z^2)$ we obtain

$$\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = 0.027730 \pm 0.000148^* \quad . \tag{23}$$

The asterisk on the error indicates that this error does not yet include the uncertainty of the p-QCD contribution which is still under investigation. Table 1 compares our results with results obtained by other authors which obtain smaller errors because they are using p-QCD in a less conservative manner.

| $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ | $\delta\Delta\alpha$ | $\delta\sin^2\Theta_f$ | $\delta M_W$ | Method | Ref. |
|---|---|---|---|---|---|
| 0.0280 | 0.00065 | 0.000232 | 12.0 | data $<$ 12. GeV | [10] |
| 0.02777 | 0.00017 | 0.000061 | 3.2 | data $<$ 1.8 GeV | [16] |
| 0.02763 | 0.00016 | 0.000057 | 3.0 | data $<$ 1.8 GeV | [18] |
| 0.027730 | 0.000148 | 0.000089 | 4.6 | Euclidean $>$ 2.5 GeV | [24] |
| 0.027426 | 0.000190 | 0.000070 | 3.6 | scaled data, pQCD 2.8-3.7, 5-$\infty$ | [14] |
| 0.027649 | 0.000214 | 0.000078 | 4.0 | same but "exclusive" | [14] |
| 0.027896 | 0.000391 | 0.000139 | 7.2 | [10] + new data CMD & BES | [13] |
| - | 0.00007 | 0.000025 | 1.3 | $\delta\sigma \lesssim 1\%$ up to $J/\psi$ | |
| - | 0.00005 | 0.000018 | 0.9 | $\delta\sigma \lesssim 1\%$ up to $\Upsilon$ | |
| world average | | 0.000160 | 22.0 | Osaka 2000 | |

Table 1: $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ and its uncertainties in different evaluations. Two entries show what can be reached by increasing the precision of cross section measurements to 1%. $\delta M_W$ in MeV.

Our procedure to evaluate $\Delta\alpha^{(5)}_{\rm had}(-M_Z^2)$ in the Euclidean region has several advantages as compared to other approaches used so far: The virtues of our analysis are the following:

- no problems with the physical threshold and resonances
- p-QCD is used only in the Euclidean region and not below 2.5 GeV. For lower scales p-QCD ceases to describe properly the functional dependence of the Adler function [25].
- no manipulation of data must be applied and we need not refer to global or even local duality. That contributions of the type of power corrections, as suggested

by the OPE, are negligible has been known for a long time. This, however, does not proof the absence of other kind of non-perturbative effects. Therefore our conservative choice of the minimum Euclidean energy seems to be necessary.

- as we shall see our non–perturbative "remainder" $\Delta\alpha^{(5)}_{\rm had}(-s_0)$ is mainly sensitive to low energy data, which changes the chances of possible future experimental improvement dramatically, as illustrated in Fig. 3.

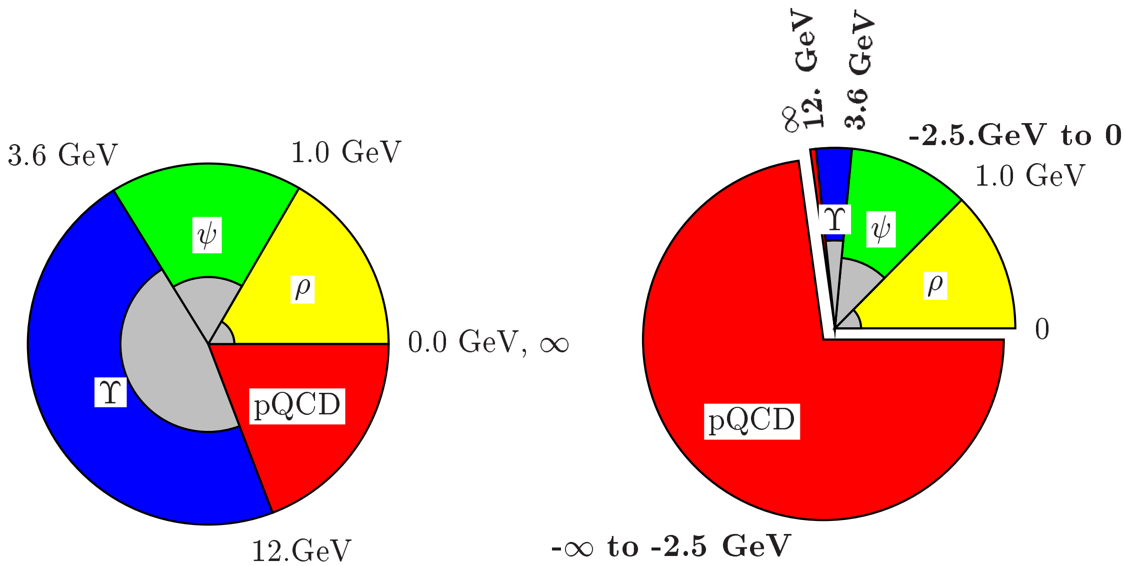



Figure 3: Comparison of the distribution of contributions and errors (shaded areas scaled up by 10) in the standard (left) and the Adler function based approach (right), respectively.

## 4 The leading hadronic contribution to $a_\mu$.

The anomalous magnetic moment of the muon $a_\mu$ provides one of the most precise tests of the quantum field theory structure of QED and indirectly also of the electroweak SM. The precision measurement of $a_\mu$ is a very specific test of the magnetic helicity flip transition $\bar{\psi}_L\,\sigma_{\mu\nu}\,F^{\mu\nu}\,\psi_R$, a dimension 5 operator which is forbidden for any species of fermions at the tree level of any renormalizable theory. In the SM it is thus a finite prediction which can be tested unambiguously to he extend that we are able to calculate it with the necessary accuracy. For the perturbative part of the SM an impressive precision has been reached. Excitingly the new experimental result from Brookhaven [1] which reached a substantial improvement in precision shows a 2.6 $\sigma$ deviation from the theoretical prediction: $\left|a_\mu^{\rm exp} - a_\mu^{\rm the}\right| = 426(165)\times 10^{-11}$. The status is illustrated in Fig. 4. We refer to Ref. [26] for a very recent review and possible implications.

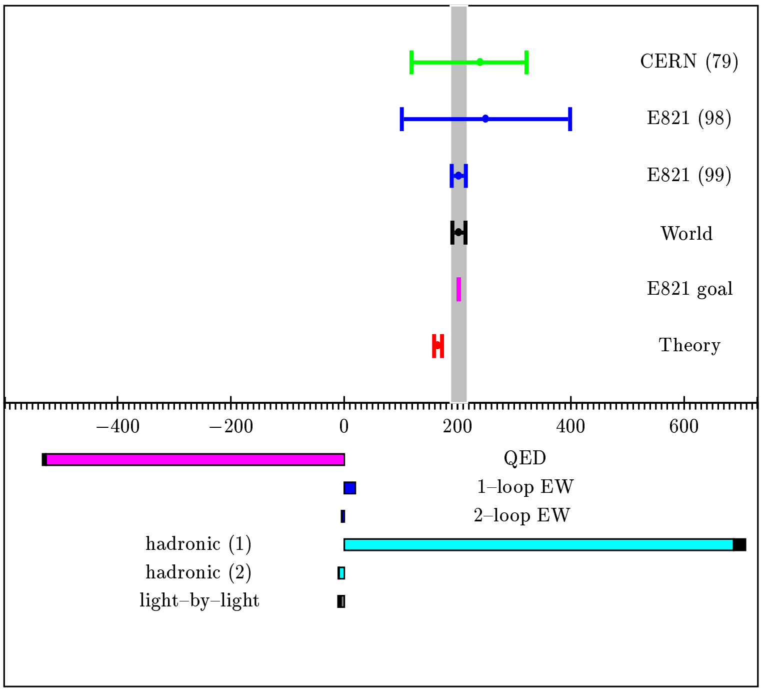
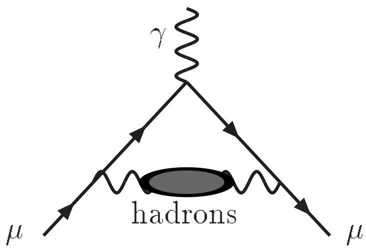



Figure 4: $a_\mu$-11659000 $\times 10^{-10}$: theory vs. experiment in the year 2001 for (g-2) of the muon. The new E821 experiment at Brookhaven reviled a 2.6 $\sigma$ deviation from the theory. The various types of SM contributions are shown in the lower part of the figure.

Again contributions from virtual creation and reabsorption of strongly interacting particles cannot be computed with the help of p-QCD and cause serious problems. Fortunately the major such contribution again enters via the photon vacuum polarization which can be calculated along the lines discussed for the effective charge. The contribution is described by the diagram



and is represented by the integral

$$a_\mu^{\rm had} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left( \int\limits_{4m_\pi^2}^{E^2_{\rm cut}} ds \, \frac{R_\gamma^{\rm data}(s) \; \hat{K}(s)}{s^2} + \int\limits_{E^2_{\rm cut}}^{\infty} ds \, \frac{R_\gamma^{\rm pQCD}(s) \; \hat{K}(s)}{s^2} \right) \tag{24}$$

which is similar to the integral (17), however with a different kernel $K(s)$ which may conveniently be written in terms of the variable

$$x = \frac{1-\beta_\mu}{1+\beta_\mu} \; , \;\; \beta_\mu = \sqrt{1-4m_\mu^2/s}$$

and is given by

$$K(s) = \frac{x^2}{2}\,(2-x^2) + \frac{(1+x^2)(1+x)^2}{x^2} \left(\ln(1+x) - x + \frac{x^2}{2}\right) + \frac{(1+x)}{(1-x)}\, x^2 \ln(x) \; . \tag{25}$$

The integral (24) is written in terms of the rescaled function

$$\hat{K}(s) = \frac{3s}{m_\mu^2} K(s)$$

which is bounded: it increases monotonically from 0.63 at threshold $s = 4m_\pi^2$ to 1 at $\infty$. Note the extra $1/s$–enhancement of contributions from low energies in $a_\mu$ as compared to $\Delta\alpha$.

A compilation of the $e^+e^-$–data in the most important low energy region is shown in Fig. 5. The relative importance of various regions is illustrated in Fig. 6. The update of the results [10], including the new data from CMD and BES yields

$$a_\mu^{\rm had} \;\; = \;\; (698.75 \pm 11.11) \times 10^{-10} \;\; . \tag{26}$$

This compares to the estimation [18]

$$a_\mu^{\rm had} \;\; = \;\; (692.4 \;\; \pm \;\; 6.2 \;\; ) \times 10^{-10}$$

and to the recent BNL $(g_\mu - 2)$ measurement [1]

$$a_\mu^{\rm exp} \;\; = \;\; (116592023 \pm \; 15.1 \;\; ) \times 10^{-10} \qquad \text{(world average)}$$

In Tab. 2 we summarize a few of the recent evaluations of the leading hadronic contributions.

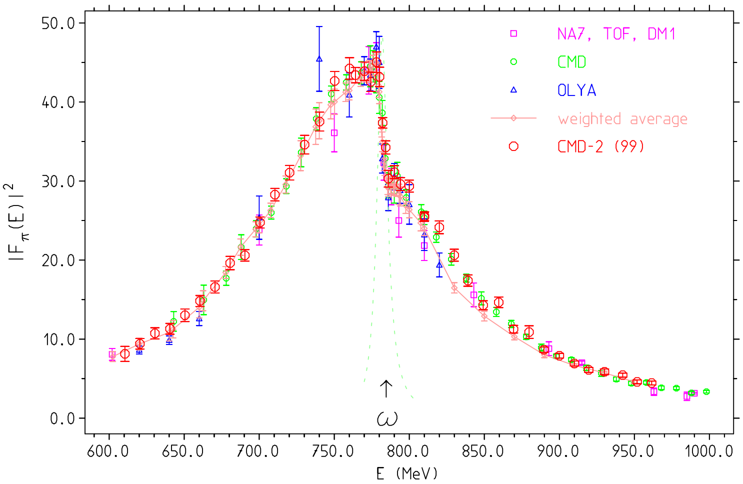



Figure 5: The dominating low energy tail is given by the channel $e^+e^- \to \pi^+\pi^-$ which forms the $\rho$–resonance. We show a compilation of the measurements of the square of the pion form factor $|F_\pi(s)|^2 = 4\,R_{\pi\pi}(s)/\beta_\pi^3$ with $\beta_\pi = (1-4m_\pi^2/s)^{1/2}$.

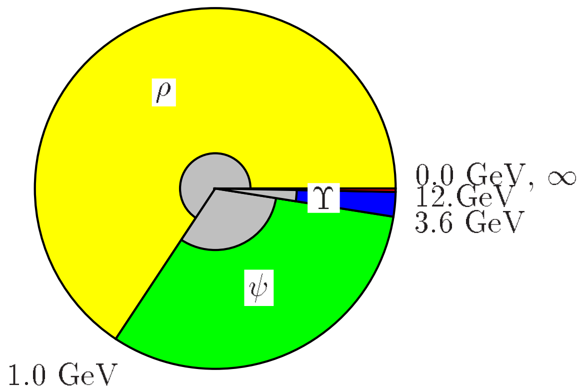



Figure 6: The distribution of contributions and errors (shaded areas scaled up by 10) for $a_\mu^{\rm had}$ .

Let me comment on Tab. 2: the first result with $\delta a_\mu \sim 156 \times 10^{-11}$ is based on $e^+e^-$–data as analyzed in Ref. [10] and confirmed in [27]. Perturbative QCD is utilized only conservatively between 5.5 and 9.6 GeV and above 11 GeV (see [9] for a discussion of the range of applicability of p-QCD). Including the $\tau$–data from ALEPH Ref. [27] finds a result with 40% improved uncertainty $\delta a_\mu \sim 94 \times 10^{-11}$ under the assumption

| $a_\mu^{\rm had}$ | $\delta a_\mu$ | Method | Ref. |
|---|---|---|---|
| 6967 | 156 | data $<$ 12. GeV + pQCD | [10,27] |
| 6987 | 111 | $e^+e^-$ data incl. CMD-2 99, BES-II 99 | [13] |
| 7011 | 94 | $e^+e^-$ + $\tau$ data $<$ 12. GeV + pQCD | [27] |
| 6951 | 75 | data $<$ 1.8 GeV and 3.3 – 5.0 GeV + pQCD | [15] |
| 6924 | 62 | in addition OPE (sum rules) | [18] |
| – | 151 | BNL $a_\mu$–experiment 2001 | [1] |
| – | 40 | BNL final goal | |

Table 2: $a_\mu^{\rm had}$ and uncertainties in units $10^{-11}$.

that iso-spin breaking is negligible. Assuming the validity of p-QCD in the extended range between 1.8 and 3.5 GeV and above 5 GeV [15] reduce the error further to $\delta a_\mu \sim 75 \times 10^{-11}$. In view of the bad quality of the data in some ranges the idea to replace them by a theoretical prediction is certainly able to lead to an improvement of the evaluation. I do not see however, how to estimate reliably a theoretical uncertainty in this approach since there are non–perturbative effects around and the assumption of local duality, i.e., $\sigma(e^+e^- \to \text{hadrons}) \simeq \sigma(e^+e^- \to \text{quarks})$ in some average sense has no a priori theoretical justification. Applying in addition the operator product expansion (OPE) and sum rules to fix the quark and gluon condensate parameters from the $e^+e^-$–data in [18] a further reduction of the error to $\delta a_\mu \sim 62 \times 10^{-11}$ was claimed. That this "improvement" is obsolete has been clearly shown by the analysis [25]. Once the full massive three loop prediction [22] for the Adler function is compared with the data it is impossible to establish any condensate effects. In the region below 2.5 GeV where their contribution gets numerically significant the perturbative expansion clearly is not reliable anymore.

A subtantial improvement of the evaluation of $a_\mu^{\rm had}$ is possible, however, by including the $\tau$–data. This has been pioneered by Ref. [27]. Here one utilizes the fact that the vector–current hadronic $\tau$–decay spectral functions are related to the iso–vector part of the $e^+e^-$–annihilation cross–section via an iso-spin rotation:

$$\tau^- \to X^- \nu_\tau \quad \leftrightarrow \quad e^+ e \to X^0$$

where $X^-$ and $X^0$ are related hadronic states. The $e^+e^-$ cross–section is then given by

$$\sigma^{I=1}_{e^+e^- \to X^0} = \frac{4\pi\alpha^2}{s} v_{1,X^-} \quad , \quad \sqrt{s} \leq M_\tau \quad .$$

The $\tau$ spectral function $v_1$ is obtained from the normalized invariant mass-squared distribution $\rho\,(X^-) \equiv (1/N_{X^-}\, dN_{X^-}/ds)$ of the $\tau$ vector channel $X^-\nu_\tau$ by

$$v_{1,X^-} = A\,\frac{B(\tau^- \to X^-\nu_\tau)}{B(\tau^- \to e^-\nu_\tau\bar{\nu}_e)}\,\rho\,(X^-)\left[\left(1-\frac{s}{M_\tau^2}\right)^2\left(1+\frac{2s}{M_\tau^2}\right)\right]^{-1}$$

where

$$A = \frac{M_\tau^2}{6|V_{ud}|^2\,(1+\delta_{\rm EW})} \tag{27}$$

with $|V_{ud}| = 0.9752 \pm 0.0007$ the CKM mixing matrix element and $\delta = 0.0194$ the electroweak radiative corrections. With the precision of the validity of CVC, this allows to improve the $I = 1$ part of the $e^+e^-$ cross–section which by itself is not a directly measurable quantity. It mainly improves the knowledge of the $\pi^+\pi^-$ channel ($\rho$–resonance contribution) which is dominating in $a_\mu^{had}$ (72%). Fig. 7 shows a compilation of the $\tau$–data from ALEPH [28], OPAL [29] and CLEO [30] in comparison to the $e^+e^-$–data.

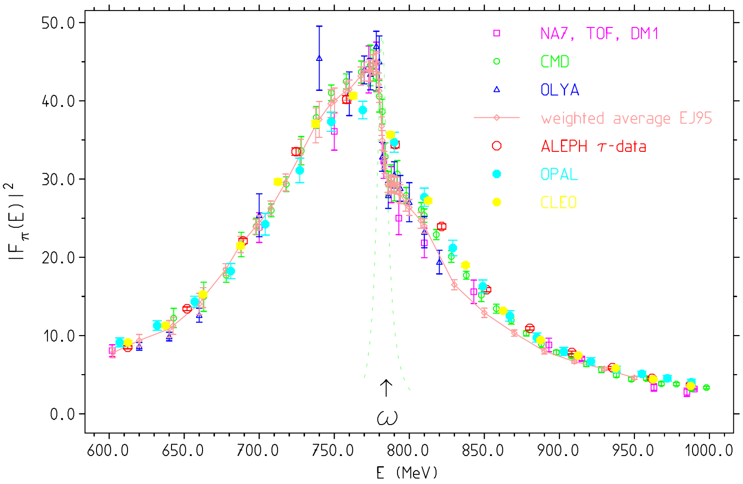



Figure 7: The square of the pion form factor $|F_\pi|^2$: a compilation of the $\tau$–decay data in comparison with the $e^+e^-$–annihilation data. The experimental cross sections are used to calculate the leading hadronic contribution to (g-2) of the muon.

## 5 Concluding Remarks

Experimental efforts to measure very precisely the total cross section $\sigma(e^+e^- \to hadrons)$ at low energies are mandatory for the future of electroweak precision physics. Taking into account recent theoretical progress, these “low energy” measurements are not only important for testing $a_\mu$ but as well for the effective fine structure constant $\alpha(M_Z)$. A real breakthrough would be possible by measuring $\sigma(e^+e^- \to hadrons)$ at 1% accuracy below the $\tau$–threshold.

Fortunately there is work in progress which can help to further reduce the uncertainties of theoretical predictions: (i) VEPP-2M Novosibirsk (CMD-2, SND): can further improve to 1.5% up to 1.4 GeV. An upgrade of the machine and the detectors is under consideration. (ii) DAΦNE Frascati (KLOE): within one year of running we expect a measurement below the $\phi$ resonance which is expected to be competitive to the Novosibirsk data. (iii) BEPC Beijing (BES): can improve a lot in the important $J/\Psi$ region. (iv) In future a "$\tau$–charm facility" tunable between 2 GeV and 3.6 GeV would settle the remaining problems essentially.

## Addendum

In Ref. [25] it has been shown how one can obtain a better control on the validity of pQCD by utilizing analyticity and looking at to problem in the $t$–channel (Euclidean field theory approach). It has been found that "data" may be safely replaced by pQCD at $\sqrt{-t} \geq 2.5$GeV. An application to the calculation of the running fine structure constant has been discussed in [24]. Here we consider the application to the calculation of $a_\mu^{had}$. Starting point is the basic integral representation

$$a_\mu^{\mathrm{h}ad} = \frac{\alpha}{\pi} \int\limits_0^\infty \frac{ds}{s} \int\limits_0^1 dx \, \frac{x^2\,(1-x)}{x^2 + (1-x)\,s/m_\mu^2} \, \frac{\alpha}{3\pi} \, R(s) \quad . \tag{28}$$

If we first integrate over x we find the well known standard representation as an integral along the cut of the vacuum polarization amplitude in the time–like region, while an interchange of the order of integrations yields an integral over the hadronic shift of the fine structure constant in the space–like domain [31]:

$$a_\mu^{\mathrm{h}ad} = \frac{\alpha}{\pi} \int\limits_0^1 dx \, (1-x) \, \Delta\alpha^{\mathrm{had}} \left(-Q^2(x)\right) \tag{29}$$

where $Q^2(x) \equiv \frac{x^2}{1-x} m_\mu^2$ is the space–like square momentum–transfer or

$$x = \frac{Q^2}{2m_\mu^2} \left( \sqrt{1 + \frac{4m_\mu^2}{Q^2}} - 1 \right) \quad .$$

In this approach we (i) calculate the Adler function from the $e^+e^-$–data and pQCD for the tail above 11 GeV, (ii) calculate the shift $\Delta\alpha^{\mathrm{had}}$ in the Euclidean region with or without an additional cut in the $t$–channel at 2.5 GeV and (iii) calculate $a_\mu^{had}$ via (29).

Alternatively, by performing a partial integration in (29) one finds

$$a_\mu^{\mathrm{h}ad} = \frac{\alpha}{\pi} m_\mu^2 \int\limits_0^1 dx\; x\,(2-x)\;\Big(D(Q^2(x))/Q^2(x)\Big) \tag{30}$$

by means of which the number of integrations may be reduced by one. The evaluation in both forms provides a good stability test of the numerical integrations involved.

Utilizing the most recent compilation of the $e^+e^-$–data we obtain the following results:

| $a_\mu^{had} \times 10^{11}$ | method |
|---|---|
| 6994.7 (104.3) | Eq. (24), pQCD above 11 GeV |
| 6974.0 (104.4) | Eqs. (18) and (29), |
| 6974.0 (104.7) | Eq. (29), pQCD above 2.5 GeV |
| 6974.0 (104.7) | Eq. (30), pQCD above 2.5 GeV |

We thus find very good stability of the results obtained by th different approaches. Not too surprisingly, as is well known the contribution to $a_\mu^{had}$ is dominated by the low energy $e^+e^-$–data below 1 GeV, here the replacement of data by pQCD does not reduce the uncertainty. The reason is hat the pQCD contribution replacing the Euclidean Adler function at $\sqrt{-t} > 2.5$GeV shows a substantial uncertainty due to the uncertainty of the charm mass $m_c(m_c) = 1.15...1.35$ GeV. The uncertainty in the strong coupling constant $\alpha_s(M_Z^2) = 0.120 \pm 0.003$ is small and is not the dominating effect. In contrast to [15] we do not obtain a reduction of the error. Of course our cut at 2.5 GeV, which we think is all we can justify, is more conservative than the 1.8 GeV in the time–like region anticipated there. Thus the best value we can obtain from presently available $e^+e^-$–data alone reads

$$a_\mu^{\mathrm{h}ad} \;=\; (697.40 \pm 10.45) \times 10^{-10} \;\; . \tag{31}$$

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